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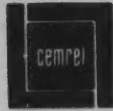
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IDENTIFIERS *Comprehensive School Mathematics Program

ABSTRACT

This guide represents the final experimental version of an extended pilot project which was conducted in the United States between 1973 and 1976. The manner of presentation and pedagogical ideas and tools are based on the works of Georges and Frederique Papy. They are recognized as having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. The CSMP curriculum follows the "spiral approach." Topics covered in the first half of the text include the minicomputer and arrow pictures, decimal numbers, negative integers, multiplication, an introduction to the rational numbers and division. The second half includes several numerical games. Worksheets for the students accompany the guide. (Author/SA)

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Comprehensive School Mathematics Program
CEMREL, Inc.

CSMP Mathematics for the Intermediate Grades Part I

Experimental Version

00701

Teacher's Guide

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
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The World of Numbers

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**The World
of
Numbers**

Comprehensive School Mathematics Program

**Mathematics
for the
Intermediate Grades
Part I**

The World of Numbers

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR MONDAY'S LESSONS

WEEK	MONDAYS	MATERIALS
1	1M M1.1: Transforming a Number and Arrow Pictures #1 p. 4	Teacher: Minicomputer set;* colored chalk Student: Minicomputer set; paper; pencil; colored pencils
2	2M M1.2: Transforming a Number and Arrow Pictures #2 p. 18	T: Minicomputer set; colored chalk S: Minicomputer set; paper; pencil; colored pencils
3	3M M2.1: Halving and Decimal Numbers p. 54	T: Colored chalk; Minicomputer set S: Pencil; paper
4	4M M3.1: Negative Integers on the Number Line and on the Minicomputer p. 92	T: Colored chalk; Minicomputer set S: Minicomputer set; pencil; paper; Worksheets N03, N04, N05
5	5M M1.3: Introduction to the Minicomputer Golf Game p. 27	T: Minicomputer set; hand-calculator S: None
6	FREE DAY	
7	6M M4.1: Multiplication Problems #1 p. 123	T: Minicomputer set; colored chalk S: Pencil; paper; Worksheet N06
8	7M M3.2: Minicomputer Golf with Positive and Negative Checkers #1 p. 99	T: Minicomputer set; colored chalk; hand-calculator S: Colored pencils; pencil; paper
9	8M M2.2: Dollars, Dimes and Pennies p. 64	T: Minicomputer set; colored chalk S: Pencil; paper; Worksheets N01, N02

* A teacher's Minicomputer set consists of 4 demonstration Minicomputer boards, 8 hooks with which to hang the boards, and a sufficient number of demonstration Minicomputer checkers. A student's Minicomputer set consists of two individual Minicomputer sheets (each containing two boards) and a sheet of individual checkers.

SUGGESTED SCHEDULE FOR MONDAY'S LESSONS

WEEK	MONDAYS	MATERIALS
10	9M M5.1: Fractions #1 p. 152	T: None S: Pencil; paper; 30-centimeter ruler
11	10M M4.2: Multiplication Problems #2 p. 133	T: Minicomputer set; colored chalk S: Pencil; paper; Worksheets N07, N08, N09
12	11M M5.2: Fractions #2 p. 159	T: None S: Pencil; paper; 30-centimeter ruler
13	12M M6.1: Division Problems #1 p. 166	T: Colored chalk S: Pencil; paper; Worksheets N13, N14, N15
14	13M M5.3: Fractions #3 p. 164	T: None S: Pencil; paper
15	14M M3.3: Minicomputer Golf with Positive and Negative Checkers #2 p. 111	T: Minicomputer set; colored chalk S: Colored pencils; pencil; paper
16	15M M2.3: Assorted Problems p. 78	T: Minicomputer set; colored chalk S: Pencil; paper; colored pencils
17	16M M4.3: Multiplication Problems #3 p. 142	T: Hand-calculator S: Pencil; paper; Worksheets N10, N11, N12
18	17M M6.2: Division Problems #2 p. 175	T: Colored chalk S: Pencil; paper; colored pencils

SUGGESTED CHRONOLOGICAL ORDER OF MONDAY'S LESSONS

Lesson One:	M1.1 - Transforming a Number and Arrow... Pictures #1	M 4*
Lesson Two:	M1.2 - Transforming a Number and Arrow Pictures #2	M 18
Lesson Three:	M2.1 - Halving and Decimal Numbers ...	M 54
Lesson Four:	M3.1 - Negative Integers on the Number Line and on the Minicomputer	M 92
Lesson Five:	M1.3 - Introduction to the Minicomputer Golf Game	M 27
Lesson Six:	M4.1 - Multiplication Problems #1 ...	M 123
Lesson Seven:	M3.2 - Minicomputer Golf with Positive and Negative Checkers #1	M 99
Lesson Eight:	M2.2 - Dollars, Dimes and Pennies ...	M 64
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Lesson Seventeen:	M6.2 - Division Problems #2	M 175

*Page numbers preceded by "M" refer to pages in the "M" part of this volume.

SUGGESTED SCHEDULE FOR FRIDAY'S LESSONS

WEEK	FRIDAYS	MATERIALS
1	1F F1.1: Tag the Arrow Pictures p. 3	Teacher: Colored chalk Student: Colored pencils; pencil; paper; Worksheets N16, N17, N18
2	2F F1.2: Match the Tags #1 p. 13	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N19, N20
3	3F F2.1: Composition of Functions #1 p. 42	T: Colored chalk; Minicomputer Set S: Pencil; paper; Worksheets N28, N29, N30
4	4F F3.1: Guess My Rule #1 p. 70	T: Colored chalk S: Pencil; paper; Worksheets N37, N38
5	5F F4.1: Ten Times #1 p. 94	T: Minicomputer Set; colored chalk S: Pencil; paper; colored pencils
6	6F F5.1: Find the Secret Number p. 117	T: Minicomputer Set S: Pencil; paper; Worksheets N42, N43
7	7F F4.2: Ten Times #2 p. 107	T: Minicomputer Set; colored chalk S: Pencil; paper; colored pencils
8	8F F2.2: Composition of Functions #2 p. 49	T: Colored chalk; Minicomputer Set S: Pencil; Worksheets N31, N32, N33
9	9F F6.1: The Relation "Is A Divisor Of" #1 p. 157	T: Colored chalk S: Pencil; paper; colored pencils

x

SUGGESTED SCHEDULE FOR FRIDAY'S LESSONS

WEEK	FRIDAYS	MATERIALS
10	10F F5.2: "Who Are We?" p. 124	T: "Who Are We?" story-workbook; colored chalk S: "Who Are We?" story-workbook; pencil
11	11F F1.3: Match the Tags #2 p. 21	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N21, N22, N23
12	FREE DAY	
13	12F F6.2: The Relation "Is A Divisor Of" #2 p. 164	T: Colored chalk S: Pencil; paper; colored pencils
14	13F F5.3: Nix; Pif and Paf p. 137	T: Colored chalk; Minicomputer Set S: Pencil; paper; Worksheets N44, N45
15	14F F3.2: Guess My Rule #2 p. 84	T: Colored chalk S: Pencil; paper; Worksheets N39, N40, N41
16	15F F1.4: Find the Missing Arrows p. 29	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N24, N25, N26, N27
17	16F F5.4: Nick and Nack p. 147	T: Colored chalk; Minicomputer Set S: Pencil; paper; Worksheets N46, N47
18	17F F2.3: Composition of Functions #3 p. 59	T: Colored chalk S: Pencil; paper; colored pencils Worksheets N34, N35, N36

SUGGESTED CHRONOLOGICAL ORDER OF FRIDAY'S LESSONS

Lesson One:	F1.1 - Tag the Arrow Pictures	...	...	...	F 3*
Lesson Two:	F1.2 - Match the Tags #1	...	...	...	F 13
Lesson Three:	F2.1 - Composition of Functions #1	...	...	...	F 42
Lesson Four:	F3.1 - Guess My Rule #1	...	...	...	F 70
Lesson Five:	F4.1 - Ten Times #1	...	...	...	F 94
Lesson Six:	F5.1 - Find the Secret Number	...	...	...	F 117
Lesson Seven:	F4.2 - Ten Times #2	...	...	...	F 106
Lesson Eight:	F2.2 - Composition of Functions #2	...	...	...	F 49
Lesson Nine:	F6.1 - The Relation "Is A Divisor of" #1	...	...	...	F 157
Lesson Ten:	F5.2 - "Who Are We?"	...	...	...	F 124
Lesson Eleven:	F1.3 - Match the Tags #2	...	...	...	F 21
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Lesson Fourteen:	F3.2 - Guess My Rule #2	...	...	...	F 84
Lesson Fifteen:	F1.4 - Find the Missing Arrows	...	...	...	F 29
Lesson Sixteen:	F5.4 - Nick and Nack	...	...	...	F 147
Lesson Seventeen:	F2.3 - Composition of Functions #3	...	...	...	F 59

* Page numbers preceded by "F" refer to pages in the "F" part of this volume.

THE WORLD OF NUMBERS

INTRODUCTION

The content of this strand is presented as two sequences of chapters; the "M" part and the "F" part. These designations simply indicate that the "M" lessons are given on Mondays and the "F" lessons on Fridays in our suggested schedule of lessons.

By this time, your students have had a rich variety of experiences with various kinds of numbers and operations and relations on them. These include exposure to the integers, both positive and negative, the decimal numbers, numerical functions (e.g., $2x$, $+3$, $+2$, -5), some rational numbers, order relations (e.g., $<$ and $>$), and the notion of multiples of a given number. They have been introduced to paper and pencil algorithms for subtraction and addition of whole numbers and decimal numbers* and have had extensive experiences in preparation for algorithms for multiplication and division.

In the present strand, these previous numerical experiences will be revisited, extended and deepened. New situations will be presented, including the introduction of several fascinating new numerical games: Minicomputer Golf (Chapter M1, Section 3); Guess My Rule (Chapter F3, Section 1); The Detective Stories (Chapter F6). Old and new concepts will be explored in the context of these situations.

The second paragraph above, describing some of your students' previous numerical experiences, simply catalogues some of the realms of the world

* Students who studied the CSMP curriculum in the third grade had some initial experience with the decimal numbers including calculating with them. Those who began with CSMP Mathematics for the Intermediate Grades, Entry Module 1 during the first semester of fourth grade will have their first introduction to the decimal numbers in Chapter 2 of the Monday part of this strand.

of numbers your students have visited. It does not indicate the travel routes nor the vehicles used to travel them.

The CSMP curriculum stresses the unity and continuity of the growth of mathematical ideas and concepts. This is reflected in numerous ways in this strand, as in the others. For example, CSMP employs the spiral approach, not pausing for mastery of each lesson but rather exposing the student to the elements of each content strand in different situations, interspersed with other strands throughout the year. It is important to recognize this approach consciously. For if you strive for mastery of each single lesson you will find yourself involved in a great deal of redundancy as the year progresses.

Further, CSMP presents the content in a situational framework. That is, a "pedagogy of situations" is employed which engages students in rich problem solving activities. These offer both the opportunity to develop necessary numerical skills and the deeper growth of understanding of mathematical ideas and concepts in the world of numbers. At the same time the situations presented encourage the student to develop patterns of logical thinking and strategies for attacking problems.

Perhaps the most important embodiments of the CSMP approach are the three nonverbal languages used throughout its elementary program: The Papy Mini-computer, the language of arrows and the language of strings. These are the vehicles which make it possible to develop the situations in which the content is presented to the student. Further, as the student grows in the CSMP program, these languages become basic investigatory tools available to explore the content.

It is hard to overstate the value of developing tools which aid in attacking problems as well as in representing situations; tools which are not confined to one area of mathematical content or to one level of the development of content. In the usual elementary curriculum, students must continually learn new language to express each new piece of mathematics that is studied.

Armed with the universally applicable languages of the CSMP program, the students grow more and more familiar with the syntax of these languages and are freed to explore new content directly in terms of these same languages rather than being forced to think of each new mathematical idea as tied to certain new language. This is certainly not to say that the CSMP students do not learn the usual descriptive language of mathematics. However, in the CSMP program the usual descriptive language is not the vehicle for learning, but only for describing the ideas.

For specific examples of the situations and the ways the CSMP languages are used in this strand we refer you to the short introductory remarks preceding each chapter and, particularly to the lessons themselves.

A WORD ABOUT STANDARD ALGORITHMS

We mentioned above that the CSMP program seeks to develop the necessary skills in the numerical domain as well as deeper understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that along with the growth of understanding in the world of numbers there must be a concomitant growth of familiarity and facility with numbers and the operations and relations on them. But facility must not be confused with understanding. They are partners in the growth of mathematical maturity, but it is extremely important that neither be sacrificed for the other. Rather, a balanced growth in each must be maintained.

In the usual elementary curriculum, paper and pencil algorithms for the four basic operations provide a prime opportunity for the abuse of the position stated above. It is undeniably important that the student eventually learns mechanical algorithms for the basic operations. However, if these algorithms are presented prematurely they stunt the student's ability to develop alternative algorithms, to do mental arithmetic, to estimate, and so on.

For example, to determine the standard name for 698×9 , it is much more efficient to proceed by multiplying 700 by 9 and then subtracting 18 than to

employ the standard algorithm. Indeed, built into this way of approaching this problem is a relatively good estimate (700×9) of the number involved.

If we insist on a mechanistic response to problems such as the one above (i.e., insist on the use of the paper and pencil algorithm) we not only might encourage inefficiency in many situations but might also inhibit the development of the flexibility necessary for problem solving. On the other hand, if we present the student a rich array of situations in which he or she interacts with numbers and their operations and let the standard algorithms grow naturally as just one facet of the student's experience, we obtain two important results. The student will gain the necessary facility with standard algorithmic procedures and at the same time retain the openness required to respond creatively to new situations in the world of numbers.

THE STANDARD MINICOMPUTER MOVES

The child who starts the CSMP curriculum in Kindergarten or Grade One meets the Minicomputer for the first time just when he or she is starting to learn about numbers and the various operations on numbers. In such a context, the Minicomputer is used as a support for the positional system of numeration and as a calculating tool that helps to strengthen the first ideas of calculation and the learning of the "basic facts." To this end, heavy emphasis is laid on the "standard" representation of numbers on the Minicomputer (i.e., the representation that uses the fewest (positive) checkers possible digit by digit) and on the "standard" moves on the Minicomputer (these are described in detail below), because these provide the most efficient means to the end of teaching the children about elementary arithmetic.

Each of the "standard" Minicomputer moves leaves unchanged the number represented on the Minicomputer. Later, in the CSMP Upper Primary Grades Program (second and third grades), the children meet moves which either increase or decrease the number represented on the Minicomputer. Your children are already quite familiar with both the "standard" and "non-standard" Minicomputer moves. Many of them have probably developed shortcuts

and may recognize the answer to a problem without putting it in "standard form."

Nonetheless, it is important for the students to realize that there is a mechanical method for performing all the "standard" moves. They require no intuition and could be done by a suitably designed machine. For this reason the "standard" moves become known in the Intermediate Grades Program as "robot" moves. In cases where the mechanical nature of the Minicomputer is to be emphasized the students are asked to make "robot" moves and are limited to the following:

$1 + 1 = 2$ $10 + 10 = 20$ $100 + 100 = 200$ $1,000 + 1,000 = 2,000$	$2 = 1 + 1$ $20 = 10 + 10$ $200 = 100 + 100$ $2,000 = 1,000 + 1,000$
two checkers on the white square of a board = one checker on the red square of the same board	one checker on the red square of a board = two checkers on the white square of the same board

$2 + 2 = 4$ $20 + 20 = 40$ $200 + 200 = 400$ $2,000 + 2,000 = 4,000$	$4 = 2 + 2$ $40 = 20 + 20$ $400 = 200 + 200$ $4,000 = 2,000 + 2,000$
two checkers on the red square of a board = one checker on the purple square of the same board	one checker on the purple square of a board = two checkers on the red square of the same board

$4 + 4 = 8$ $40 + 40 = 80$ $400 + 400 = 800$ $4,000 + 4,000 = 8,000$	$8 = 4 + 4$ $80 = 40 + 40$ $800 = 400 + 400$ $8,000 = 4,000 + 4,000$
two checkers on the purple square of a board = one checker on the brown square of the same board	one checker on the brown square of a board = two checkers on the purple square of the same board

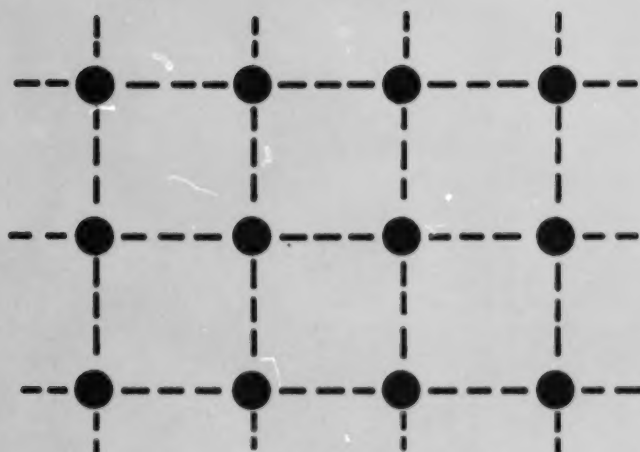
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$8 + 2 = 10$ $80 + 20 = 100$ $800 + 200 = 1,000$ $8,000 + 2,000 = 10,000$	$10 = 8 + 2$ $100 = 80 + 20$ $1,000 = 800 + 200$ $10,000 = 8,000 + 2,000$
one checker on each of the brown and red squares of a board = one checker on the white square of the next board to the left	one checker on the white square of a board = one checker on each of the brown and red squares of the next board to the right[*]

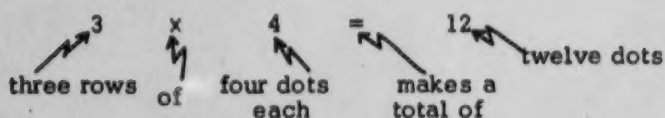
NOTE ON THE ORDER OF MULTIPLICATION

Should the function "multiply by 2" be denoted by "2 x" or "x 2"? Does it matter? The answer to this last question is, "Yes and no." It all depends on your viewpoint and on the context in which you want to use it. In order to appreciate this fact a little better let us go back to the very beginning: multiplication of nonnegative whole numbers. How does the following array of 12 dots appear to you?

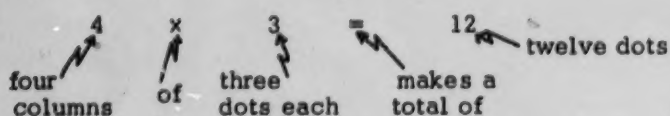
^{*} This provides the link that motivates the introduction of the decimal numbers. In particular, we have that one checker on the white square of the ones' board equals one checker on each of the brown and red squares of a special board (the tenths' board – or as it's referred to in the lessons, the "dimes'" – board) located to the right of the ones' board and separated from it by a special green barrier.



If you see three rows of four (the red dashed lines), this fact can be recorded as:



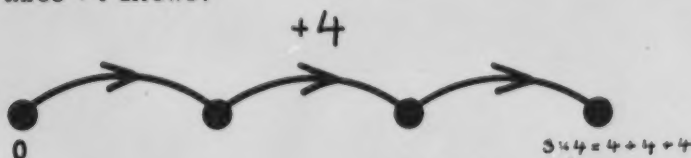
On the other hand, if you see four columns of three (the blue dashed lines), this fact can be recorded as:



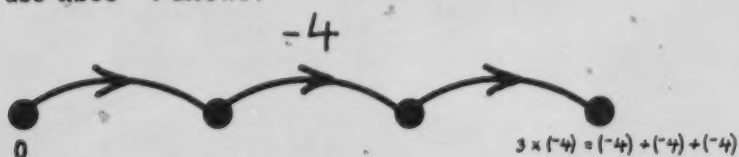
In that they describe two different perceptions, " 4×3 " and " 3×4 " are obviously very different. But, equally obviously, " 4×3 " and " 3×4 " are both names for the same number, namely 12. So, in this case, it makes not the slightest difference if we denote "multiply by 3" by " $\times 3$ " or " $3 \times$ ": if you write down the product of some nonnegative integer and 3, you will end up naming the same number whichever of these two notations you use. This is true for any nonnegative integer, n : " $\times n$ " and " $n \times$ " may be used indifferently to denote "multiply by n ," provided the number you are multiplying by n is also a nonnegative integer.

Problems don't arise until negative integers and (non-integer) decimal numbers put in their appearance, the reason being that it's a bit more difficult to find a model for multiplication when negative integers and (non-integer) decimal numbers are around. What would it mean, for example, to have an array of dots with six rows each containing negative two dots, or, even worse, negative three rows each containing negative five dots? What about 3.72 rows each containing seven dots? Fortunately, it is possible to find such a model which allows the calculation of at least some products involving negative integers or (non-integer) decimal numbers; but, unfortunately, we lose the advantage of the array-of-dots model which makes it evident that the order of multiplication does not influence the answer. We call upon the language of arrows:

- 3×4 is the number obtained if you start from 0 and use three $+4$ arrows.



- $3 \times (-4)$ is the number obtained if you start from 0 and use three -4 arrows.



This model has the advantage that it not only "pictures" the product, but it also provides a means of calculating that product: just calculate the names of all the dots until you reach the one in which you are interested. The disadvantage is that it doesn't make any sense to speak of a negative number of arrows starting from 0. Thus, by the end of this semester, strictly speaking, your children will only have been introduced to products in which the first number is a whole number (the second number can be either positive or

negative or zero and can even be a (non-integer) decimal number). This has implications for notation:

- (i) As noted above, if only nonnegative integers are involved, you can use " $2 \times$ " or " $\times 2$ " as the fancy takes you (we have scattered both of these notations throughout this semester's curriculum).
- (ii) If a negative integer (say, -4) is being multiplied by 2 (say), then, strictly speaking, you should denote this " $2 \times (-4)$ " and not " -4×2 ." This is because " 2×-4 " carries the sense of " $(-4) + (-4)$," whereas " -4×2 " has no natural connotation at this stage. In other words, strictly speaking, you should use " $2 \times$ " when negative integers and (non-integer) decimal numbers are involved, never " $\times 2$." We have observed this rule in the Teacher's Guide for this semester.

We have emphasized the phrase "strictly speaking" above because this is not something you should make into a big issue with your children. In fact, it would probably be best to avoid all mention of this subject in your classroom if at all possible. If any of your children denote "times" functions one way when strictly speaking they should do it the other way, simply accept it without comment. There could be a variety of reasons why some of your children might do this; among them are:

- (i) To conform to a pattern: all the other types of functions are usually written with the operation symbol on the left (-2 ; $\div 3$; $+6$).
- (ii) They have generalized the results from the case of nonnegative integers and find it perfectly natural to believe that the order of multiplication makes no difference in the case of any integers.

Our concern in this note has been to make you aware of some of the underpinnings of this issue so that, in the very unlikely event that a problem arises in your class, you will have the necessary background information to understand why it might have occurred, thus perhaps pointing the way to a means of getting out of the difficulty.

Of course, you really have nothing to worry about: if your children infringe any of the above "rules," what they do will not be wrong (because, after all, it really doesn't make any difference in which order you multiply two integers), it's just that, at this stage, what they do will be unfounded.

A WORD ON THE USE OF HAND CALCULATORS^{*}

The hand calculator provides another powerful tool for the investigation of problems and situations in the world of numbers. Appropriately, this tool is a natural extension of the Papy Minicomputer, since its internal structure is an electronic simulation of the Minicomputer. This analogy need not be consciously acknowledged by the student but serves to indicate that the hand calculator is indeed a natural tool in the CSMP program.

Two hand calculators are included in each complete classroom set of CSMP materials for Part I of the Intermediate Grades Program. In the first half of Part I they are used for Minicomputer Golf (M1, Section 3, and M2, Section 2.) In the second half of Part I, there are more extensive opportunities for the use of the hand calculators. In addition, the supplementary material for the slow learner^{*} relies heavily on the use of hand calculators.

We are aware that it is unrealistic, indeed premature at this time, to develop extensive classroom materials which require that every student in the class have a hand calculator. We are firmly convinced through our own experiences with children using hand calculators, that the hand calculator is a potent pedagogical tool when used properly. The time is not far off when its presence in the classroom will be as commonplace as the pencil. For the present, CSMP will continue to investigate the use of the hand calculator in the classroom, particularly with the slow learner, and will develop situations designed for its use.

^{*} See General Introduction, Section 2(f) for a more detailed discussion on hand-calculators.

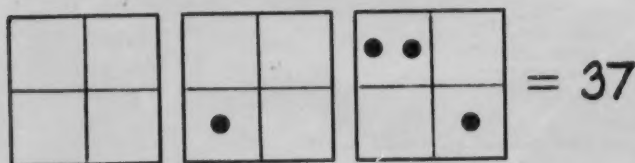
^{*} See "Mathematical Play Therapy." The introduction to that book provides suggestions for its use.

M

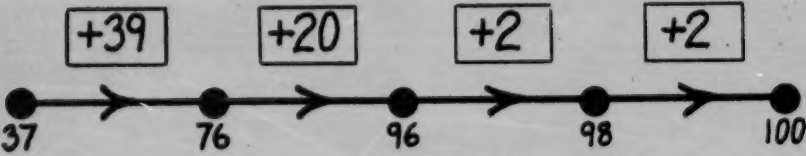
THE MINICOMPUTER AND ARROW PICTURES

In earlier grades of the CSMP program, the Minicomputer is used as a support for the positional system of numeration; for calculations; for number patterns; for modeling the basic operations involving whole numbers, integers, and decimal numbers. These uses of the Minicomputer are continued in other chapters of the World of Numbers strand. However, in this first chapter the Minicomputer is used to present challenging numerical situations which (in conjunction with the language of arrows) involve sequences of "moves" under various constraints (i.e., one checker at a time; the shortest number of moves, etc.) to achieve pre-set goals. The language of arrows serves to specify moves on the Minicomputer in some situations and to record them in others.

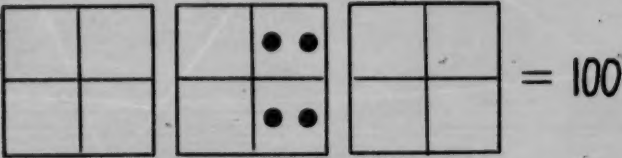
Some of the activities in this chapter are similar to those in Part II of the CSMP Upper Primary Grades Program, some extend and deepen these experiences and some are completely new. All the activities serve the dual purposes of reinforcing numerical skills and encouraging strategic thinking. The following is an example of one sort of situation presented in this chapter:



Starting at 37, as shown on the Minicomputer above, the students attempt to "transform" 37 to 100 by moving exactly one checker at a time. The sequence of moves is recorded by using an arrow diagram with the appropriate calculation made for each move. Of course, there are many possible solutions. One of them is:



which yields the configuration:



At this point the activity is already a rich and purposeful one. But the teacher then puts the checkers back to their original configuration for 37 and says, "Try to solve the problem using a smaller number of moves, if you can!" The students are now in a marvelously open, yet finely focused situation and in possession of all the tools needed to solve the problem. You will find one possible solution by consulting the first section of the chapter.

Minicomputer Golf is introduced in the last section of the chapter. This game, and variations of it, will be used repeatedly. You are referred to Section 1.3 of this chapter for details and examples of the game.

CAPSULE LESSON SUMMARIES

Section 1.1: Decode various configurations on the Minicomputer, some of which include negative checkers. Gradually build a configuration for 500, practicing mental arithmetic. Draw an arrow diagram with preset arrow designations and challenge the students to follow the directions of the arrow diagram on their Minicomputers, moving exactly one checker for each arrow. Continue with the activity described in the Introduction to this chapter and other similar situations.

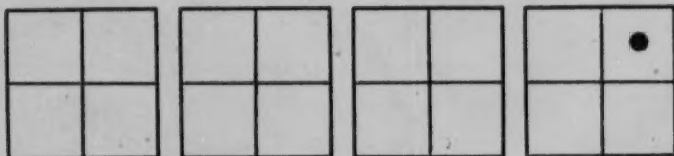
Section 1.2: Continue to develop situations similar to those of Section 1.1.

Section 1.3: Introduce the Minicomputer Golf game.

1.1 Transforming a Number and Arrow Pictures #1

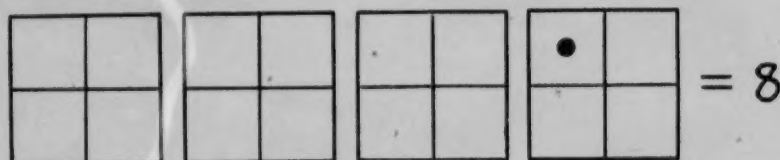
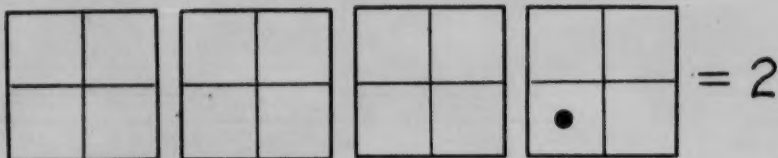
Exercise 1:^{*}

Display four Minicomputer boards, put one checker on the 4-square and take the students through a brief review in the following manner:



T: What is this number? (Answer: 4)

Ask similar questions for



^{*}If you used CSMP Mathematics for the Intermediate Grades, Entry Module I during the first semester of the fourth grade, you should omit Exercise 1, which is a review of the language of the Minicomputer. If most of your students are CSMP veterans, there will probably be a few pupils who did not study the CSMP curriculum last year. For special help with these students, see the appendix to this chapter.

Remove the checker and let several students take turns putting the following numbers (with the smallest number of checkers possible) on the ones' board:

1, 2, 4, 8, 3, 5, 9, 6 and 7.

Using one checker and starting on the 1-square of the ones' board, quickly move the checker to the square of the same color on the tens' board and similarly to the hundreds' board and the thousands' board, letting a different student identify each new number. (E.g., 1; 10; 100; 1,000)

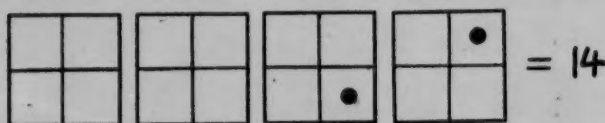
Do a similar exercise with:

2; 20; 200; 2,000

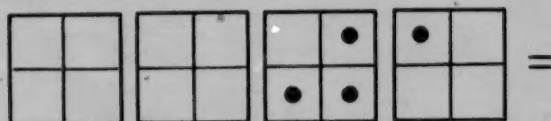
4; 40; 400; 4,000

8; 80; 800; 8,000

T: Who can put 14 on the Minicomputer with exactly two checkers?



One by one, put these four checkers on the Minicomputer and ask:



T: What is this number?

S: 78

Write "78" beneath the Minicomputer boards, lining up the digits with the boards. Do a similar exercise with two or three numbers, for instance,

1M

	•

•	•

	•
•	

 $= 136$

1 3 6

•	

•	
	•

	•
	•

 $= 895$

8 9 5

•	

	•
•	•

	•

 $= 2,740$

2 7 4 0

Remove all the checkers and ask:

T: Who can put 7 on the Minicomputer using both positive and negative checkers?

•	
	⊖

 $= 7$

Accept any correct solution.

Ask for several other numbers to be shown using both positive and negative checkers, such as:

5; 16; 37; 83; 746; 2,519; etc.

In all cases, accept any correct answer. Whenever you feel your students are ready, proceed with the next exercise.

Exercise 2:

Put numbers on the board* (with positive and negative checkers) and have students decode them. For instance,

•	•

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	•

•	
	•

	•
•	•

 $= 3,597$

•	

	•
•	•

	•
•	•

 $= 5,076$

	•

⊖	

 $= 98$

•	

	•
	•

⊖	⊖

 $= 247$

	•

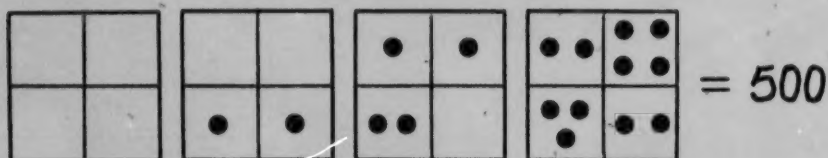
	⊖
	•

 $= -3$

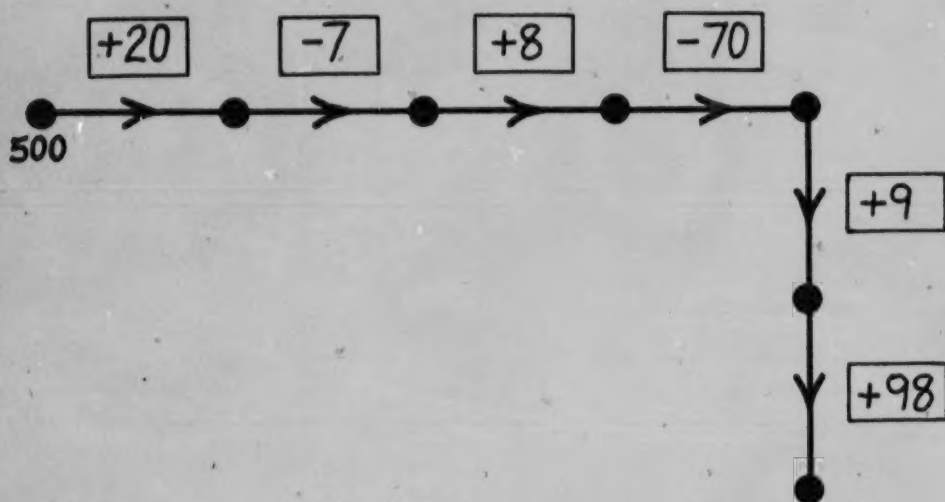
*"On the board" here, as well as with all exercises involving the Mini-computer, is an abbreviation for "On the demonstration Mini-computer."

Exercise 3:

Slowly and progressively put all these checkers on the board and have the students calculate this number^{*}:



Draw this arrow picture on the board:



^{*}There are many variations for the presentation of this type of exercise, which is repeated several times in the lessons of this chapter. One way is the following: put four checkers on the 4-square and let a student calculate the number (16). Add two checkers on the 1-square and call on another student (18). Add three checkers on the 2-square (24). Add two checkers on the 8-square (40). Continue in this way until all the checkers are on the board. Be sure that when you're finished the number on the board is 500.

T: The first arrow starts from the number on the Minicomputer (500) and asks you to make the number 20 bigger by moving exactly one checker from one square to another square.
How can you do it?

S: By moving one checker from the 20 - square to the 40 - square.

Have a student make the move.

T: What number is on the Minicomputer now?

S: 520

Write "520" by the dot at the end of the +20 arrow.

T: What do we have to do next?

S: Make the number seven smaller.

T: How can that be done?

S: Move a checker from the 8 - square to the 1 - square.

Have a student make the move.

T: What number is on the Minicomputer now?

S: $520 - 7 = 513$.

Write "513" by the dot at the end of the -7 arrow. Follow similar steps for the other arrows:

$+8$: move a checker from the 2 - square to the 10 - square.
The number now on the Minicomputer is 521.

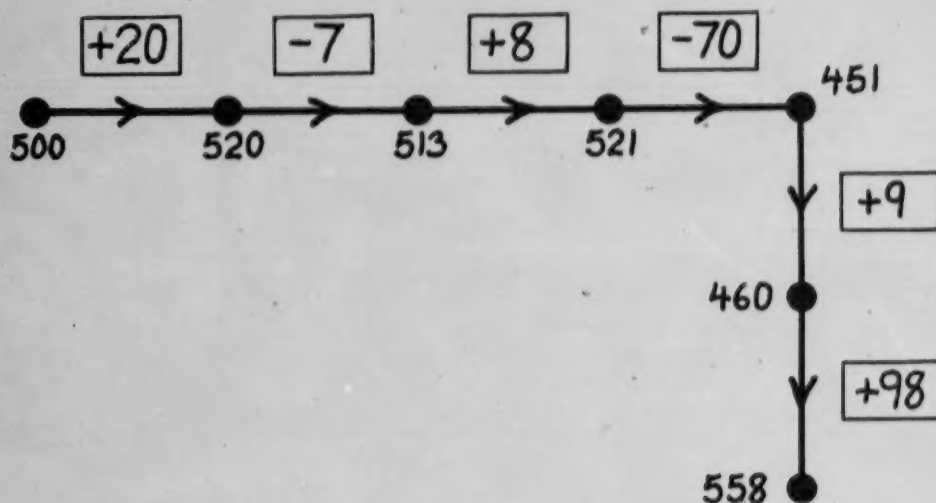
1M

-70 : move a checker from the 80 - square to the 10 - square.
The number now on the Minicomputer is 451.

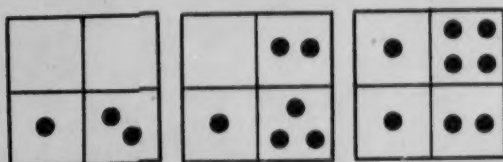
+9 : move a checker from the 1 - square to the 10 - square.
The number now on the Minicomputer is 460.

+98 : move a checker from the 2 - square to the 100 - square.
The number now on the Minicomputer is 558.

The arrow picture has been progressively completed



and the checkers on the Minicomputer moved so as to obtain the following configuration:



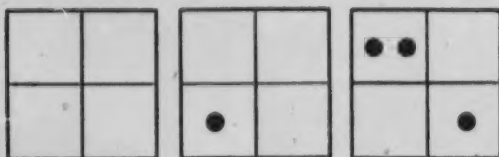
T: What number should be on the Minicomputer now?

S: 558

Have the students calculate the number on the Minicomputer to see if it's correct.

Exercise 4:

Put on the board:



T: What is this number?

S: 37

T: By moving some of these checkers from one square to another square, we have to transform 37 to 100.
Who would like to make the first move?

Let's suppose that a student suggests moving a checker from the 1 - square to the 40 - square.*

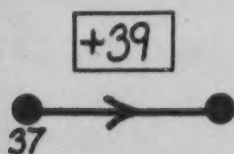
T: How much bigger is the number now?

S: 39 bigger.

*From here on you will have to adjust your lesson to the reactions of your students.

1M

Draw this arrow on the board:



T: What number is now on the Minicomputer?

S: $37 + 39 = 76$.

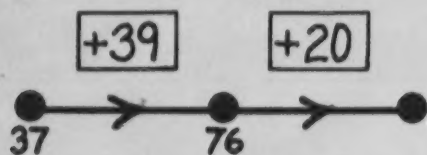
T: Next move?

Let's suppose that a student moves a checker from the 20 - square to the 40 - square.

T: How much bigger is the number?

S: 20 bigger.

Continue the picture on the board by drawing a +20 arrow:



T: What number is on the Minicomputer?

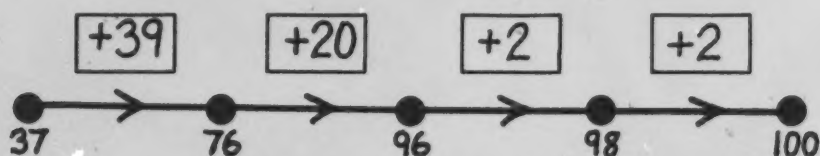
S: 96

Write "96" by the dot at the end of the +20 arrow.

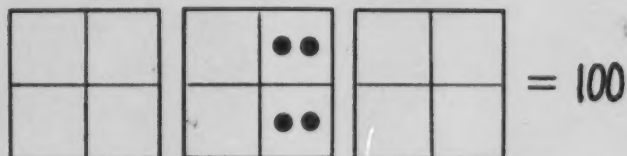
T: Next move?

The students will probably suggest moving successively the two checkers from the 8 - square to the 10 - square.

Have them complete the arrow picture

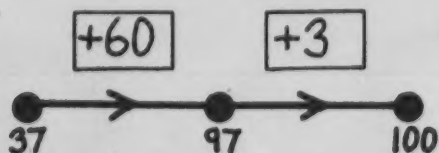


and calculate the number on the Minicomputer to see if it's correct.



Put the checkers in their original position (for 37) and ask the students to try to solve the problem with a smaller number of moves if possible.

You will probably get:



$+60$: move a checker from the 20-square to the 80-square.

$+3$: move a checker from the 1-square to the 4-square.

In this case, the configuration on the Minicomputer is:

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet\bullet & \bullet \\ \hline & \\ \hline \end{array} = 100$$

Exercise 5:

Write these three problems on the board:

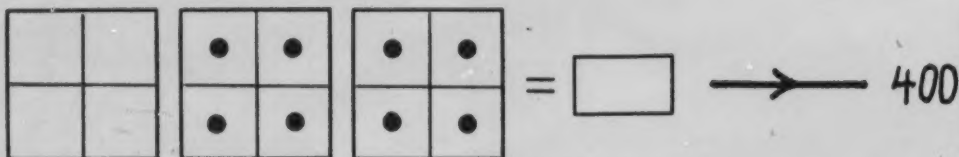
Problem *

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet\bullet \\ \hline \end{array} = \boxed{} \rightarrow 25$$

Problem **

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \bullet\bullet \\ \hline \bullet & \bullet\bullet\bullet \\ \hline \end{array} = \boxed{} \rightarrow 100$$

Problem ***



The problem is two-fold:

- 1) Write in the box the number that is on the Minicomputer.
- 11) By moving exactly one checker from one square to another square transform the number in the box to the number at the end of the arrow. Try to do this in as few a number of moves as possible.

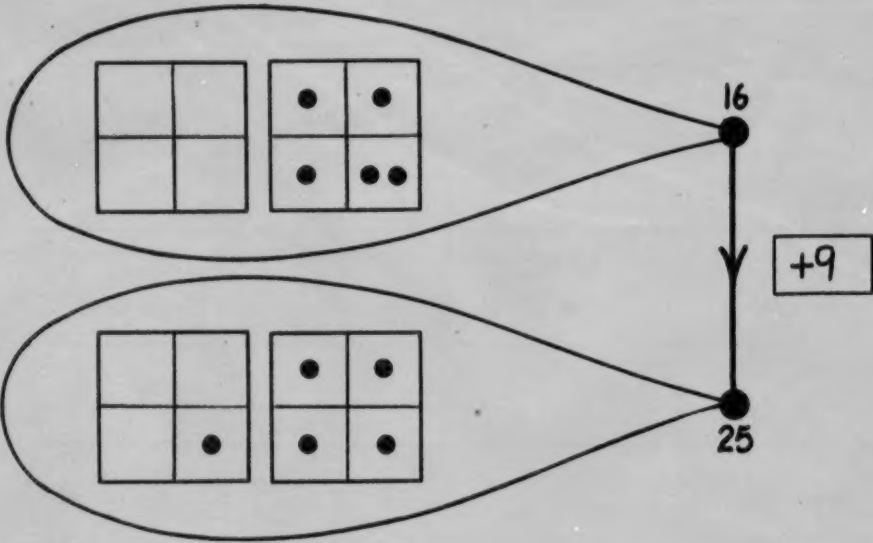
Ask the students to solve as many problems as they can with the help of their desk Minicomputers and to draw an arrow picture to present their solutions. Have them work in pairs (or alone if they prefer).

After a while, have a pair of students present their solution for one of these problems to the whole group. One of the students has to move the checkers while the other draws the arrow picture.

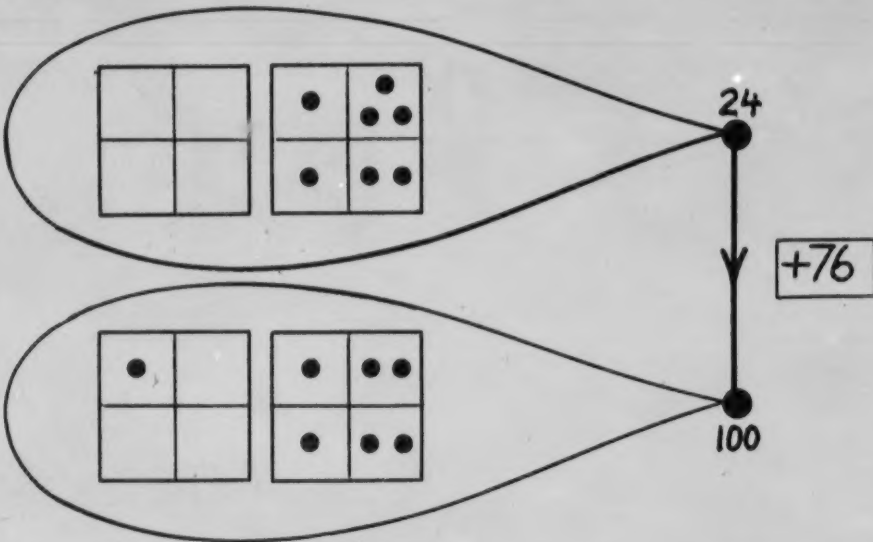
For each of the three problems, here is a minimal solution:

1M

Problem *

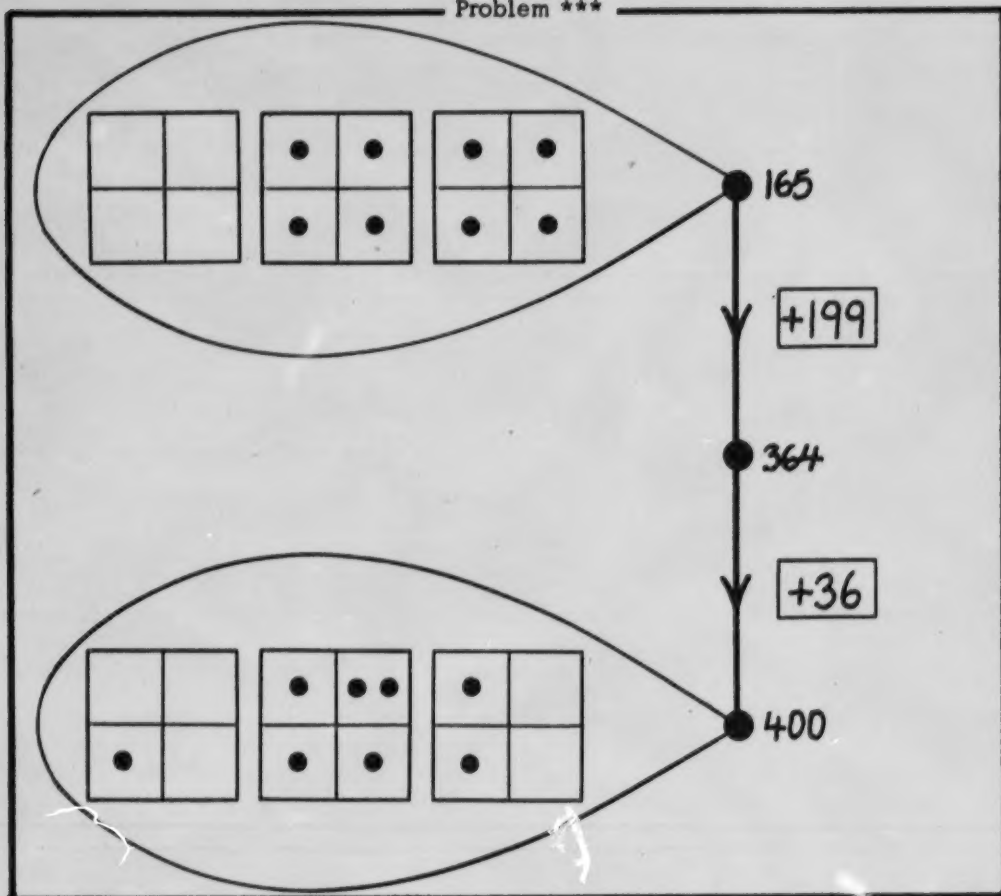


Problem **



16

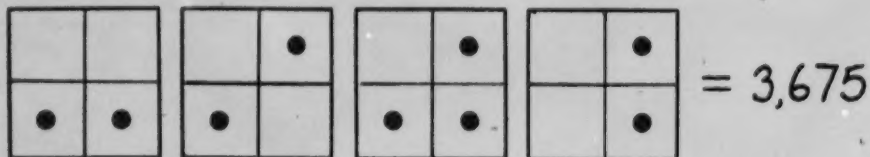
Problem ***



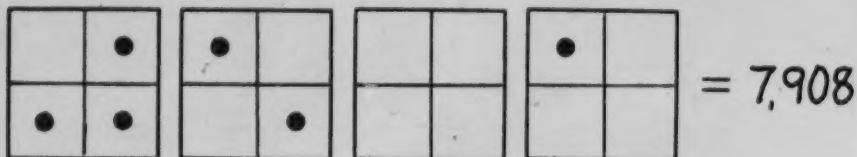
1.2 Transforming a Number and Arrow Pictures #2

Exercise 1:

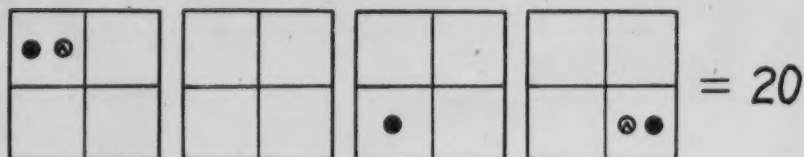
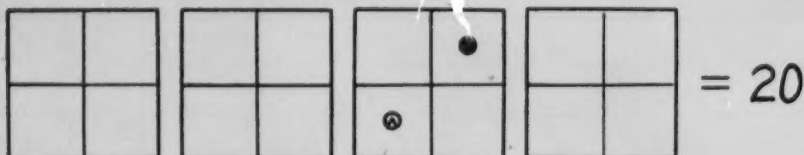
Ask the students to put the number 3,675 on their desk Minicomputers.
Have one of them put the number on the board:

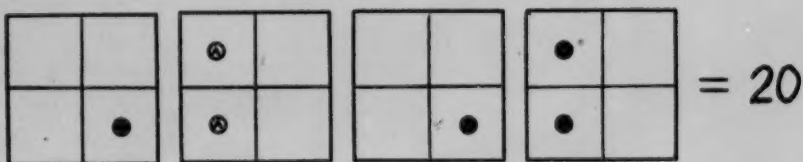
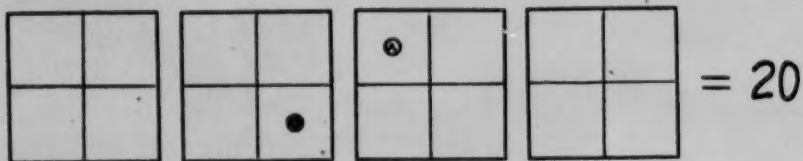


Do a similar exercise for 7,908:



Ask the students to put 20 on their desk Minicomputers by using positive and negative checkers. There are many answers for this problem. Have students put some of them on the board. For instance,

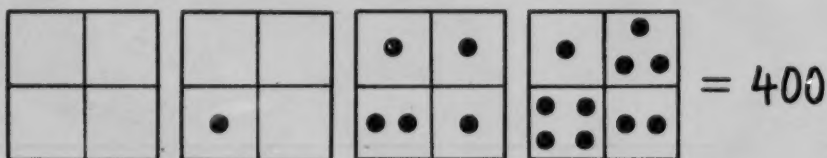




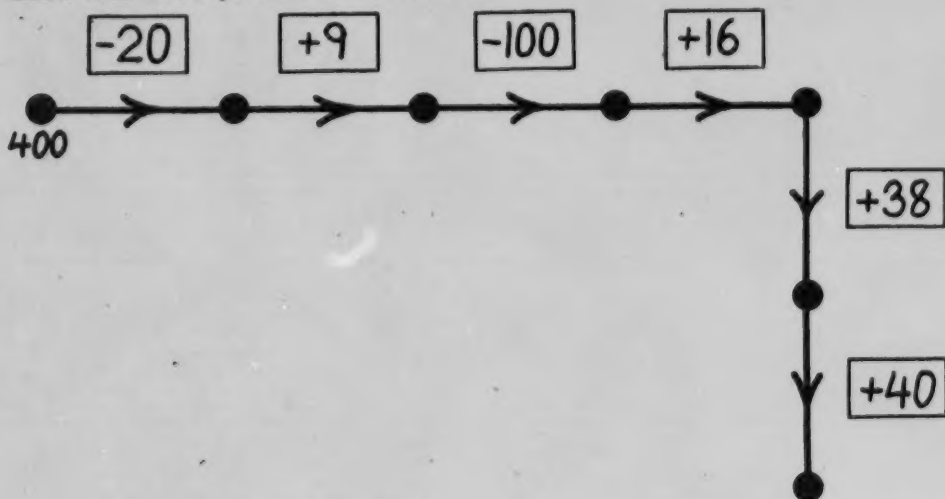
and so on, and so on. Your students can be very creative in this situation.

Exercise 2:

Slowly and progressively put all these checkers on the board and have the students calculate this number:



Draw this arrow picture on the board:



T: The first arrow starts from the number on the Minicomputer (400) and asks you to make the number 20 smaller by moving exactly one checker from one square to another square.
How can you do it?

S: By moving one checker from the 40 - square to the 20 - square.

Let a student make the move.

T: What number is now on the Minicomputer?

S: $400 - 20 = 380$.

Write "380" by the dot at the end of the -20 arrow.

T: What do we have to do next?

S: Make the number nine bigger.

Let a student move a checker from the 1 - square to the 10 - square.

T: What number is now on the Minicomputer?

S: $380 + 9 = 389$.

Write "389" by the dot at the end of the +9 arrow.

Follow similar steps for the other arrows:

-100

: move a checker from the 200 - square to the 100 - square.
The number now on the Minicomputer is 289.

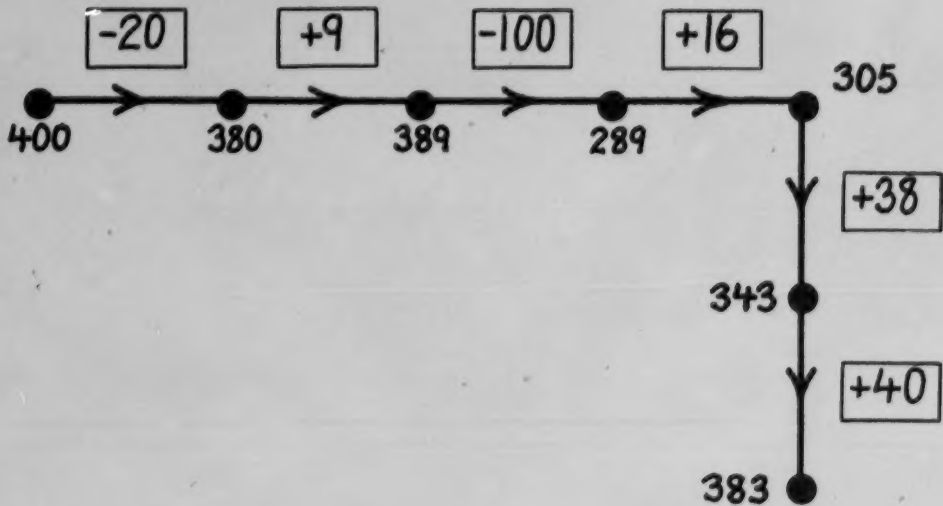
+16

: move a checker from the 4 - square to the 20 - square.
The number now on the Minicomputer is 305.

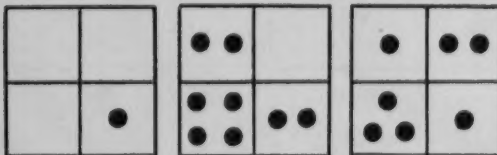
+38 : move a checker from the 2 - square to the 40 - square.
The number now on the Minicomputer is 343.

+40 : move a checker from the 40 - square to the 80 - square.
The number now on the Minicomputer is 383.

The arrow picture has been progressively completed



and the checkers moved to this configuration:



T: What number should be on the Minicomputer now?

S: 383

Have the students calculate the number on the Minicomputer to see if it's correct.

T: By moving some more checkers, let's try to get 400 again on the Minicomputer.

How much more do we need to go from 383 to 400?

Write on the board:

$$383 + \square = 400$$

Let several students whisper the answer (17) in your ear and have one of them write "17" in the frame:

$$383 + \boxed{17} = 400$$

T: Can we make the number 17 bigger with exactly one move?
(Answer: No)

T: Can we make the number 17 bigger with exactly two moves?

There are several possibilities. For instance,

1. $\boxed{+16}$: move a checker from the 4 - square to the 20 - square.

- $\boxed{+1}$: move a checker from the 1 - square to the 2 - square.

or 2. $\boxed{+20}$: move a checker from the 20 - square to the 40 - square.

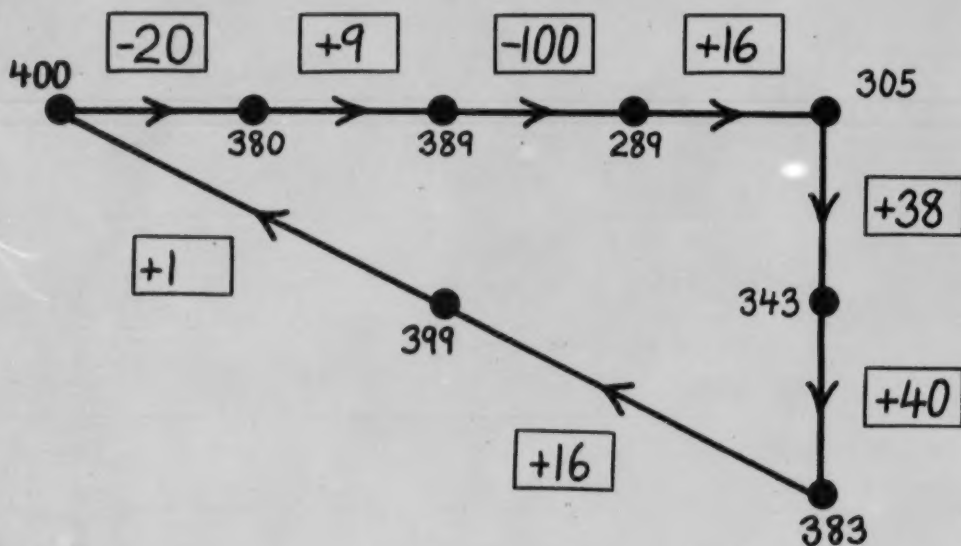
$\boxed{-3}$: move a checker from the 4 - square to the 1 - square.

or 3. $\boxed{+10}$: move a checker from the 10 - square to the 20 - square.

$\boxed{+7}$: move a checker from the 1 - square to the 8 - square.

Follow the suggestions of your students. Have one of them make the two moves and another one complete the arrow picture.

If the first solution is chosen, your complete arrow picture will be:



2M

and the configuration on the Minicomputer:

$$\begin{array}{|c|c|} \hline & \\ \hline & \bullet \\ \hline \end{array}
 \begin{array}{|c|c|} \hline \bullet \bullet & \\ \hline \bullet \bullet \bullet & \bullet \bullet \\ \hline \end{array}
 \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet \bullet & \\ \hline \bullet \bullet & \\ \hline \end{array} = 400$$

Exercise 3:

Write these problems^{*} on the board:

Problem *

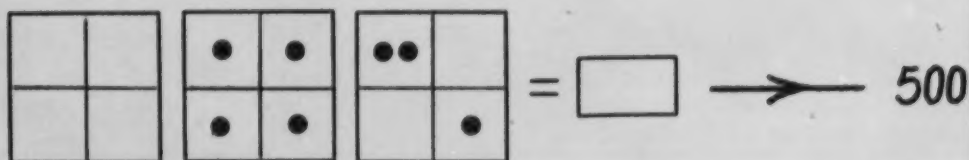
$$\begin{array}{|c|c|} \hline & \\ \hline & \bullet \\ \hline \end{array}
 \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} = \boxed{} \rightarrow 15$$

Problem **

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline & \bullet \\ \hline \end{array}
 \begin{array}{|c|c|} \hline \bullet \bullet & \bullet \\ \hline \bullet & \bullet \bullet \\ \hline \end{array} = \boxed{} \rightarrow 100$$

^{*}The directions for these problems are the same as for those in Exercise 5 of Section 1.1.

Problem ***

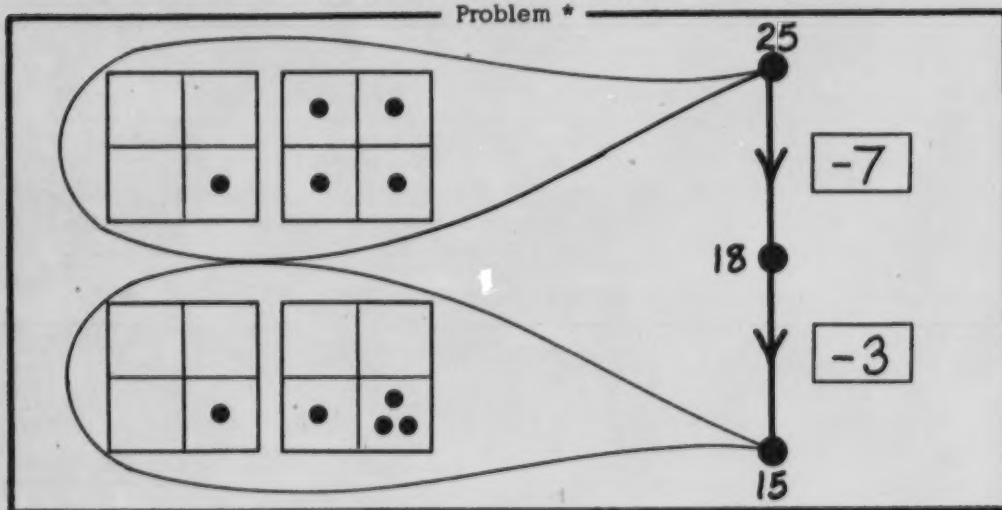


Ask the students to solve as many problems as they can with the help of their desk Minicomputers and to draw an arrow picture to present their solutions. Have them work in pairs (or alone if they prefer).

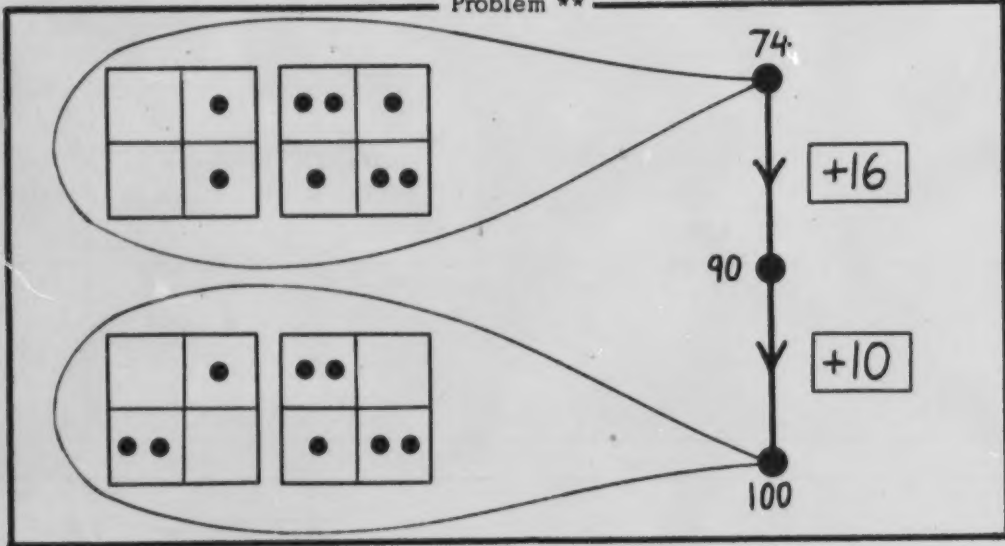
After a while, have a pair of students present their solution for one of these problems to the whole group. One of the students has to move the checkers while the other draws the arrow picture.

For each of the three problems, we give below a minimal solution:

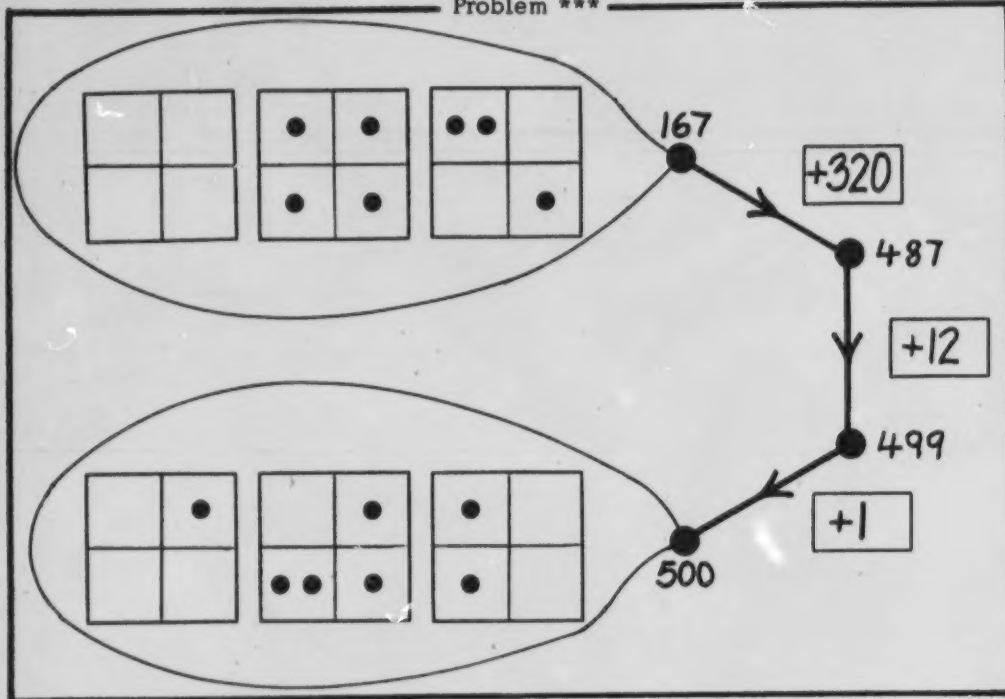
Problem *



Problem **



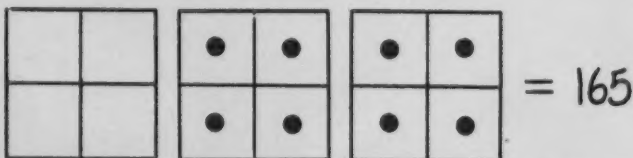
Problem ***



1.3 Introduction to the Minicomputer Golf Game

T: Today we will play a game with the Minicomputer called "Minicomputer Golf."

Put these checkers on the Minicomputer and ask the students to calculate this number:



Divide your class into three groups. One group will be Team A, another group will be Team B and the third group (which may consist of only one or two students) will receive hand-calculators and will have to check progressively the results of the game.

Write on the board:

Goal: 400

Team A:



Team B:



T: We are starting from 165 and we have to reach 400. Students from Team A and from Team B will play in turns. Each time, they can move one checker from any square to any other square. The number may be increased or decreased. The first team to reach the goal will be the winner.

Who would like to play first for Team A?

Let's suppose that the first player on Team A moves a checker from the 40-square to the 80-square. (We reproduce here the results of a game played in one of our fourth grade development classes.)

T: Is the number on the Minicomputer bigger or smaller?

S: Bigger.

T: How much bigger?

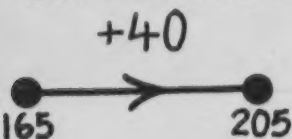
S: 40 bigger.

T: What number is now on the Minicomputer?

S: $165 + 40 = 205$.

This answer is to be checked by the third group with the hand-calculators.

Draw this red arrow on the board:



Now a player from Team B takes a turn. He or she moves a checker from the 8 - square to the 20 - square.

T: Is the number on the Minicomputer bigger or smaller?

S: Bigger.

T: How much bigger?

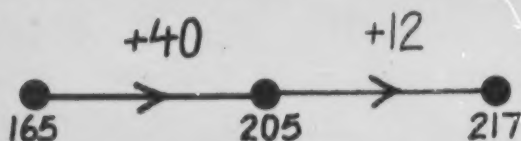
S: 12 bigger.

T: What number is on the Minicomputer now?

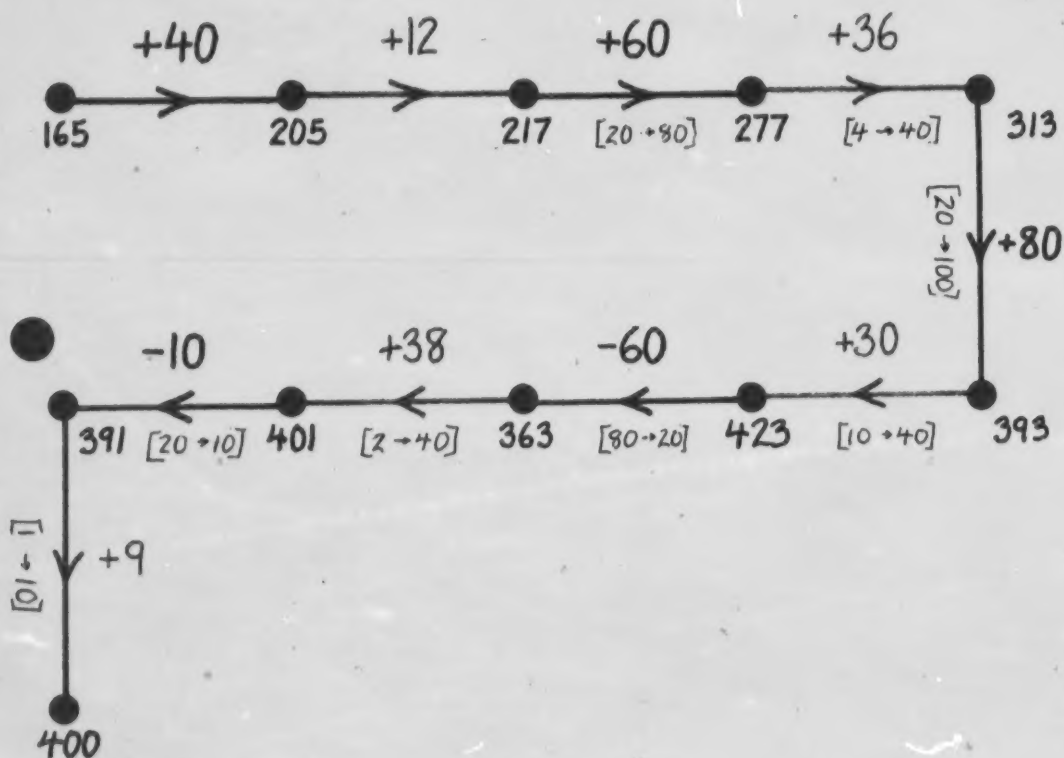
S: $205 + 12 = 217$.

Have those with hand-calculators verify this answer.

Draw this blue arrow on the board:

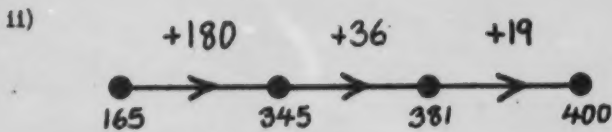
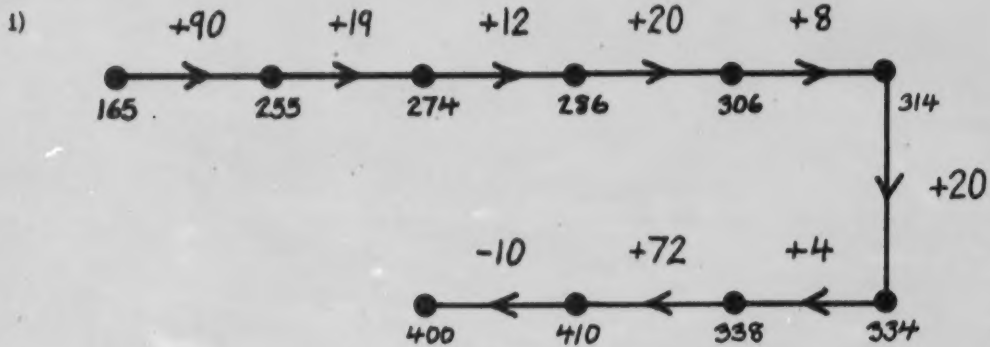


Follow similar steps for the next players, alternately from Team A and Team B. The results of the game are described by this red - blue road. Team B was the winner.

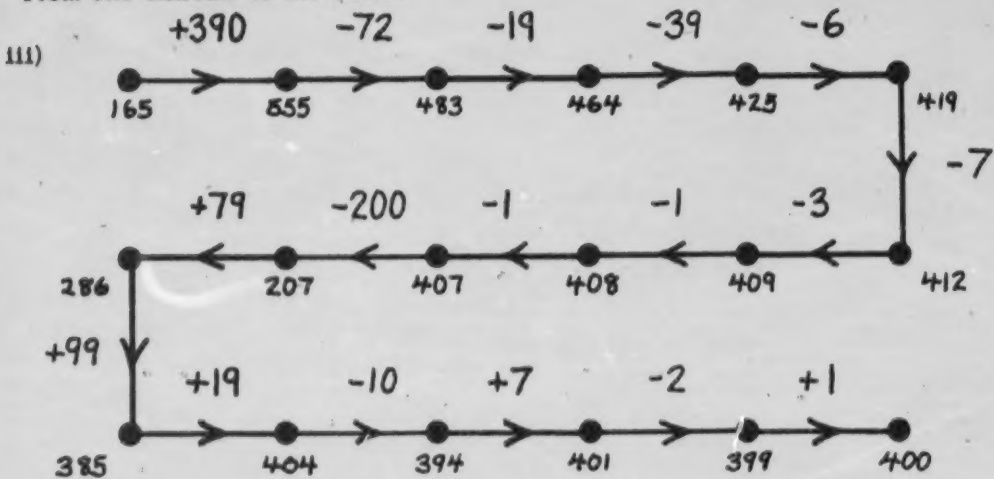


Exchange the roles between the three groups and play two more games from 165 to 400.

To show you how many possibilities any one game may have, we include below the results of three other games played in our fourth grade development classes. (The record of the actual moves on the Minicomputer have been omitted in order not to complicate the picture.)



From one extreme to the other!



If your students are still interested, you could end the lesson with a game from 165 to 1,000.

Variations of this game are presented in Sections 3.2 and 3.3 of Chapter 3 (Negative Integers) and in Chapter 2 of "Mathematical Play Therapy," a supplementary book of small group activities which is included among your materials.

APPENDIX

In this section we include two introductory lessons with the Minicomputer which are designed for students who did not have the CSMP curriculum last year and who need to be taught the basic language of the Minicomputer. In your materials we have included two copies of the storybook TWO BY TWO and we suggest you give your new pupils a chance to read them before you teach the first lesson.

Obviously, two lessons will not be enough to bring these new students to the level of most of the others. However, remember that children learn as they watch and listen to others. You can stimulate this kind of learning as well as give more opportunities for active responses by using short exercises (especially those that have been popular in a particular lesson) on the Minicomputer occasionally, when you have a few moments from time to time.

No doubt you will have students who will be pleased to share their knowledge of the Minicomputer language with their new classmates and in a few weeks you will discover that many of the new students are on a par with their peers.

In most cases it would be best to schedule a few special sessions with just your new students for the purpose of teaching them the introductory lessons. However, if this is not possible, or if you should have many pupils who would benefit by a more thorough review after the long summer vacation, then make these collective lessons which precede the first scheduled Minicomputer lesson and readjust your schedule accordingly. In this case, feel free to increase the difficulty of any exercise which seems too easy for the majority of your students and to move along at a pace which best fits the numerical ability of your class.

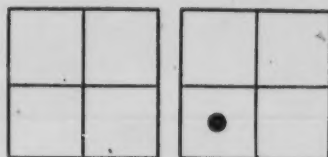
Introductory Lesson #1

Display two Minicomputer boards and have a third one ready. Put one checker on the 4-square.

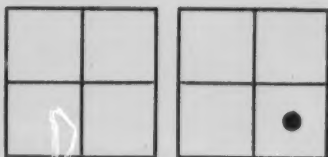
T: If you think for a moment about the storybook TWO BY TWO which I gave you to read, then you probably remember this is the number 4.



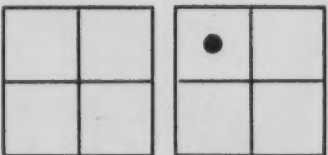
T: Who can put the number 2 on the Minicomputer?



the number 1?

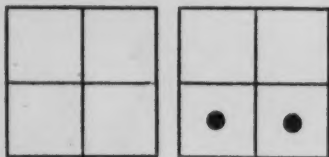


the number 8?



Quickly review the placement of the checker for 1, 2, 4, 8, asking different students to answer.

T: I'll give you two checkers. Can you put 3 on the Minicomputer?



Again with two checkers ... the number 5?

The number 6?

The number 9?

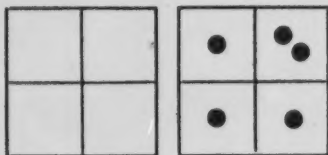
How many checkers do you need to put the number 7 on the Minicomputer?

S: Three.

(Have someone do it.)^{*}

Exercise 1:

Put some checkers on the board, for instance,



^{*} Of course, the number 7 can be written on the Minicomputer with 3, 4, 5, 6 or 7 positive checkers (in fact any number of checkers if we include boards to the right of the ones' board.) This exercise encourages the students to think about how to place the checkers before actually doing it. Accept any correct answer.

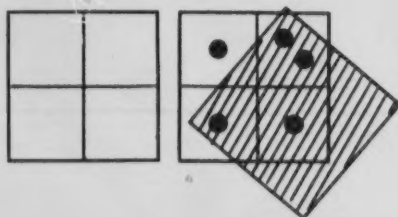
T: What is this number?

NOTE: You should put the checkers very gradually on the Minicomputer; for example ,

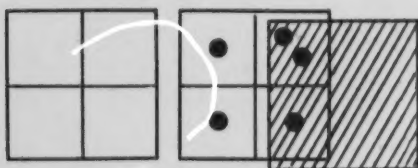
first one checker on the 8-square,
then one checker on the 2-square,
then two checkers on the 4-square,
then one checker on the 1-square.

This gives your pupils a chance to figure out mentally what number it is.

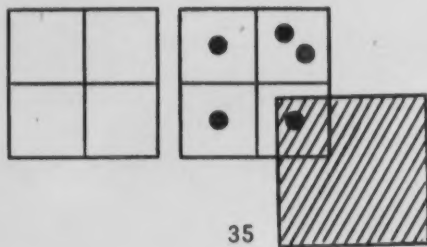
No explanation is necessary if everybody gets the right number. If some-one is wrong, you might use the following procedure: hide some of the checkers for a moment, or focus the class's attention on specific checkers:



S: Eight



S: and two is ten



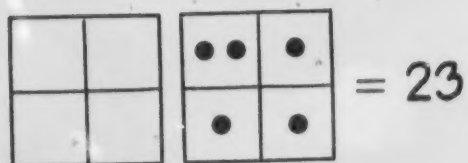
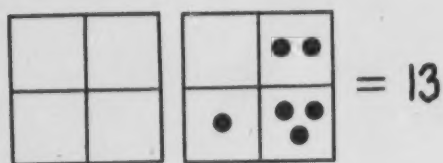
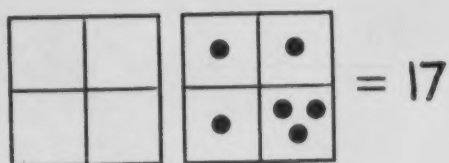
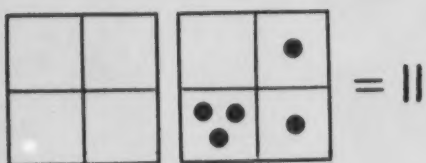
35

S: and eight (or four and four) is 18.

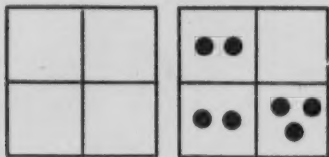
T: And one more is?

S: 19.

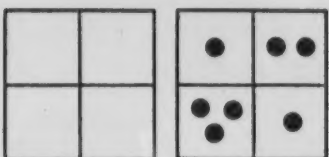
Some other examples:



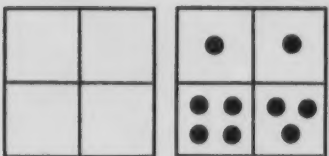
It is interesting to present the same number several times, each time with a different number of checkers. For instance (using only ones' board), 23 can be represented as



or



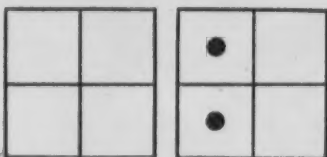
or



and so on.

Exercise 2:

Put on the Minicomputer:

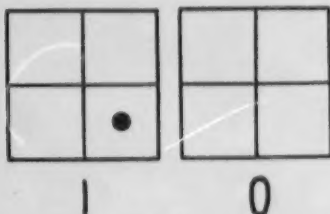


T: What is this number?

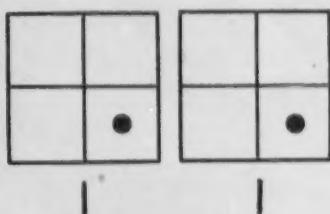
S: 10.

Make the robot move* yourself and say the words, " $8 + 2 = 10$."

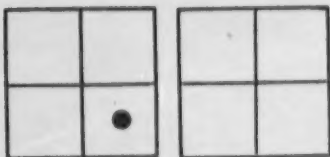
Also write the numerals under the boards:



T: This is a way to write 10 with just one checker. I'll give you one more checker. Can you write 11?



Back to



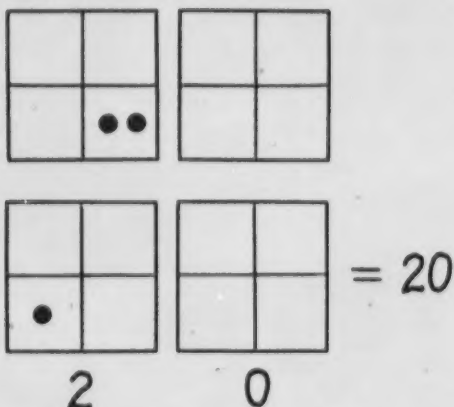
T: Again one more checker. Can you write 12? 14? 17?
How many checkers do you need?
Can you write 18? 19? 20?

* See STANDARD MINICOMPUTER MOVES in the introduction to the World of Numbers strand.

The question "How many checkers do you need to write 17?" has many answers. For the "standard configuration" you need 4 checkers, but it is possible to write 17 with 3 checkers; e.g., " $8 + 8 + 1$."

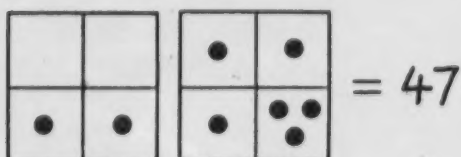
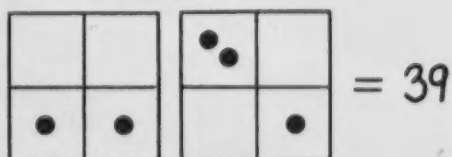
Put two checkers on the tens' board, make the robot move and say,

" $10 + 10 = 20$."

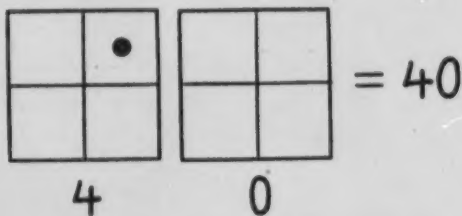


T: How can you write 26? 29? 30?

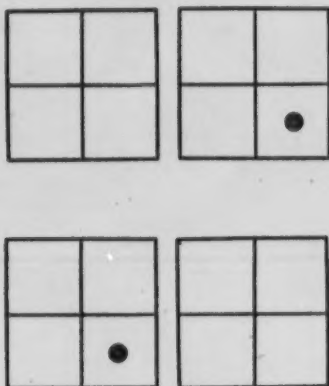
Put up the following configurations, one after the other, and each time ask for the number being displayed:



For the last configuration make the robot move and say, "20 + 20 = 40."



Put one checker on the 1-square, and very quickly move it to the 10-square like this: (Have the students chant the numbers being displayed at each step.)



Repeat with 2 and 20; 4 and 40; 8 and 80.

T: Who would like to write 49 on the Minicomputer? 57? 96? 35?

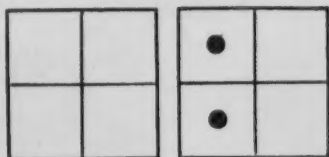
Call on volunteers to put these numbers on the demonstration Minicomputer.

Exercise 3:

Give a child five checkers. Let her or him put them anywhere on the Minicomputer (display only two boards). The other children then try to determine the number. You may want them to write their answers on a sheet of paper.

Exercise 4:

Put on the Minicomputer:

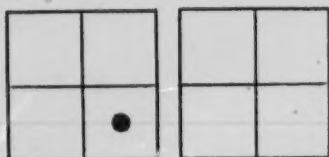


T: What is this number?

S: 10.

T: Write 10 with one checker.

S:



Now change the configuration to:



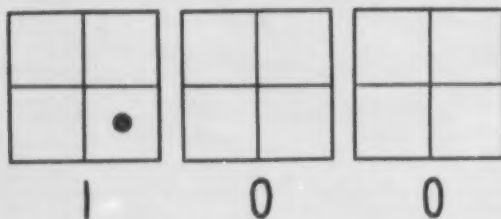
T: What is this number?

S: 100.

T: How do you know?

S: $80 + 20 = 100$.

Put up a third Minicomputer board and one checker as follows:



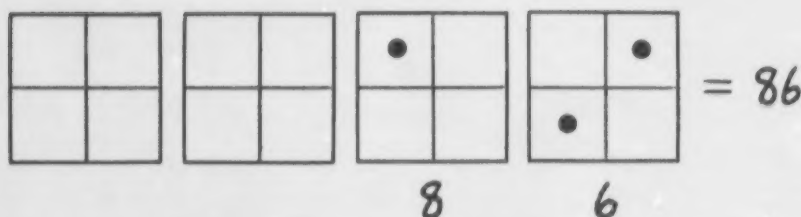
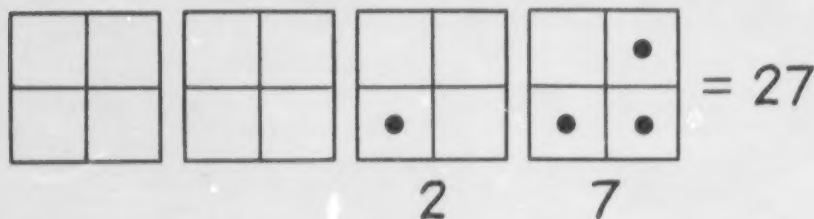
T: This is another way to write 100, with just one checker.
Who can write 105 on the Minicomputer? 126? 150? 179? 247?
463? 821? 795? 358? and so on.

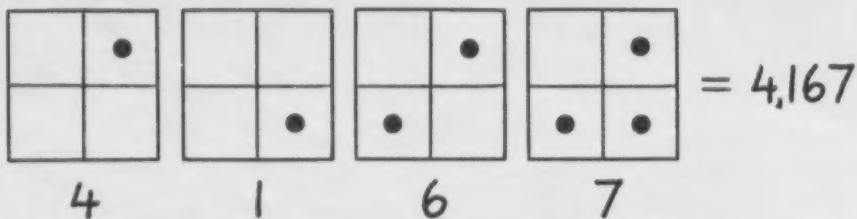
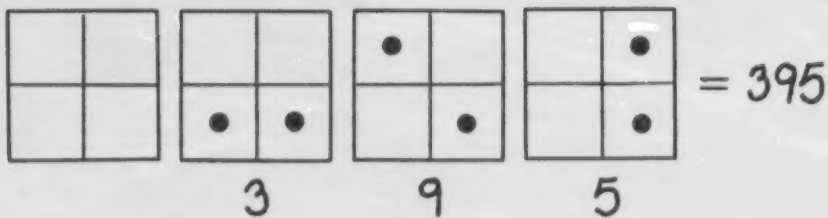
Have different students put these numbers on the demonstration Minicomputer with three boards displayed.

Introductory Lesson #2

Display four Minicomputer boards. Very quickly:

T: What are these numbers? (After the students have answered correctly, write the answers on the board in the two ways shown below.)





T: Write the following numbers on the Minicomputer: 83; 57; 416; 8,239.

Move checkers quickly from one board to the other to review:

1; 10; 100; 1,000

8; 80; 800; 8,000

2; 20; 200; 2,000

3; 30; 300; 3,000

4; 40; 400; 4,000

5; 50; 500; 5,000

and so on.

Exercise 1:

Put on the board:

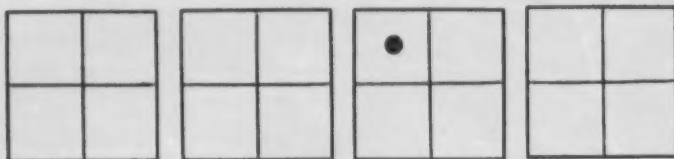


T: What is this number?

S: 80.

T: Can you write 80 with just one checker?

S:



If necessary you should repeat the robot move very clearly and say, "40 + 40 = 80 and 80 = 40 + 40."

Similarly put up the following configurations and make the appropriate robot moves saying them out loud as you do:

$$400 + 400 = 800 \text{ and } 800 = 400 + 400$$

$$1,000 + 1,000 = 2,000 \text{ and } 2,000 = 1,000 + 1,000$$

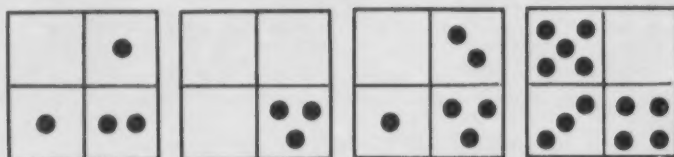
$$2,000 + 2,000 = 4,000 \text{ and } 4,000 = 2,000 + 2,000$$

$$80 + 20 = 100 \text{ and } 100 = 80 + 20$$

and so on.

Exercise 2:

Put on the Minicomputer:



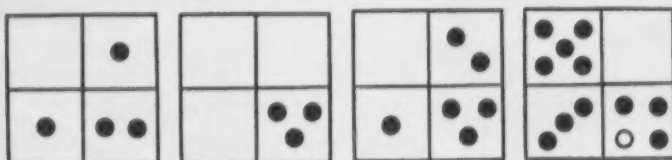
T: I have put a number on the Minicomputer.

For the moment, don't try to find out what number it is.

I will do something to this number by moving or removing or adding some checkers.

Each time, you are to tell me if I get a larger number, a smaller number or the same number.

Replace one of the checkers by a checker of another color:



T: What do you think?

S: The same.

T: Why?

S: Because all you did was change the color of a checker.

Move one checker from the 20-square to the 200-square.

S: Larger.

T: Why?

S: Because you replaced 20 by 200.

T: How much larger?

S: 180.

Return the checkers to their original positions.

T: Move one checker so that the new number is smaller.

Have a child do this. Ask another student if it is smaller and if so, how much smaller. Then return the checkers to the original positions.

Remove two checkers from the 40-square and put one checker on the 80-square.

S: The same.

T: Why?

S: $40 + 40 = 80$.

Return the checkers to their original positions.

Remove one checker from the 8-square and put two checkers on the 4-square.

S: The same.

T: Why?

S: $8 = 4 + 4$.

Remove two checkers from the 1,000-square and put one on the 4,000-square.

S: Larger.

T: Why?

S: Because $1,000 + 1,000 = 2,000$ and you replaced 2,000 with 4,000.

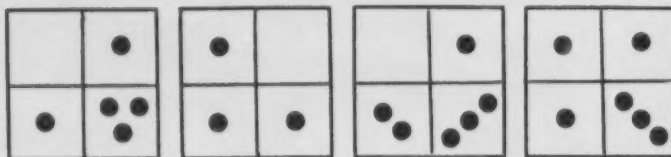
Continue this with variations as long as it seems interesting and helpful.

Exercise 3:

To one student:

T: I'll give you 20 checkers. Put them on the Minicomputer wherever you want.

Suppose the student puts them as follows:



T: Is this number larger than 2,000? How can you tell?
 Is this number larger than 5,000? How can you tell?
 Is this number larger than 8,000? How can you tell?
 Is this number larger than 10,000? How can you tell?

(To answer these questions your pupils may point to the checkers, or make some robot moves.)

T: Is it easy to calculate this number?
 What do you suggest?

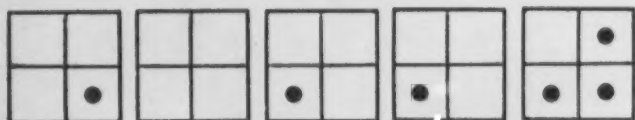
Most likely somebody will suggest making robot moves to get the number represented with the smallest number of checkers possible (e.g., on the ones' board: $1 + 1 = 2$; $8 + 2 = 10$; and so on through the other boards.)

Have the children say, " $1 + 1 = 2$," " $8 + 2 = 10$ " and so on as the robot moves are made.

After the first few moves, ask, "Did he or she change the number?" (No)

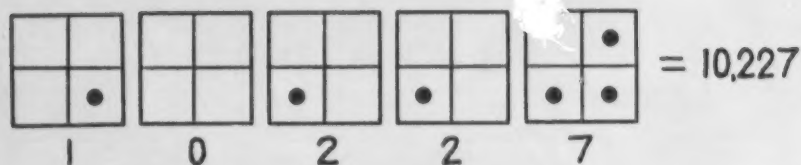
Have several children take turns to make the rest of the moves (and if this is a collective lesson, be sure to include the new students in this activity.)

Finally: (adding a 5th board will be necessary in the process)



Read the number with the children.

Write:



Repeat this kind of exercise two or three times if your students seem interested.

Exercise 4:

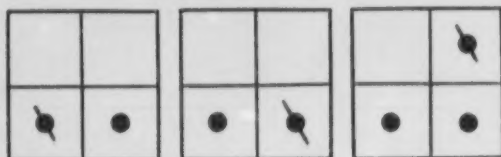
Write on the board:

$$337 - 214 =$$

Accept several "guesses" and list them on the board.

T: Put 337 on the Minicomputer. Take away 214.

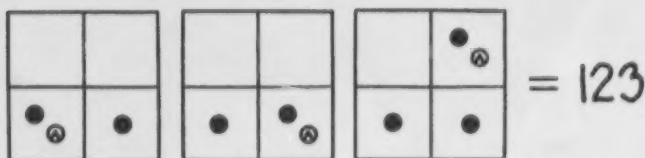
Have a student remove the checkers for 214



and write on the board:

$$337 - 214 = 123$$

T: I can show you the same calculation on the Minicomputer by adding negative checkers instead of taking positive checkers away:



Write on the board:

$$337 + \widehat{214} = 123$$

T: Whenever you have a positive and a negative checker on the same square, they cancel each other, so you only have to consider the checkers which are left. In our example above we have $200 + \widehat{200} = 0$; $10 + \widehat{10} = 0$; $4 + \widehat{4} = 0$.

Repeat this exercise both ways with:

$$553 - 412 = 141$$

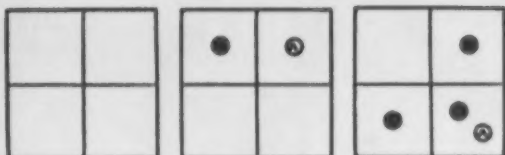
$$998 - 673 = 325$$

$$687 - 561 = 126$$

(NOTE: You may need to make some backward robot moves first.)

Exercise 5:

Put on the Minicomputer:



T: What is this number?

S: $87 - 41 = 46$.

Do a similar exercise with these calculations:

$$59 - 42 = 17$$

$$693 - 372 = 321$$

$$958 - 836 = 122$$

Exercise 6:

Display three Minicomputer boards.

T: Who can put 8 on the Minicomputer using both positive and negative checkers?

Many solutions are possible, so accept any one that is correct. For example, $9 + \hat{1}$; $10 + \hat{2}$; $20 + \hat{10} + \hat{2}$; $100 + \hat{80} + \hat{10} + \hat{2}$; and so on.

Do a similar exercise with 2; 17; 23; 30; 96; 145; 582, and so on.

2

DECIMAL NUMBERS

Decimal numbers are introduced in Part II of the CSMP Upper Primary Grades Program, motivated by our decimal system of money. This chapter reintroduces decimal numbers and proceeds to extend the students' experiences to a considerable extent. Included among these extensions are: location of points for decimal numbers on the number line, multiplication of decimal numbers by 10 and some problems involving the addition of decimal numbers.

CAPSULE LESSON SUMMARIES

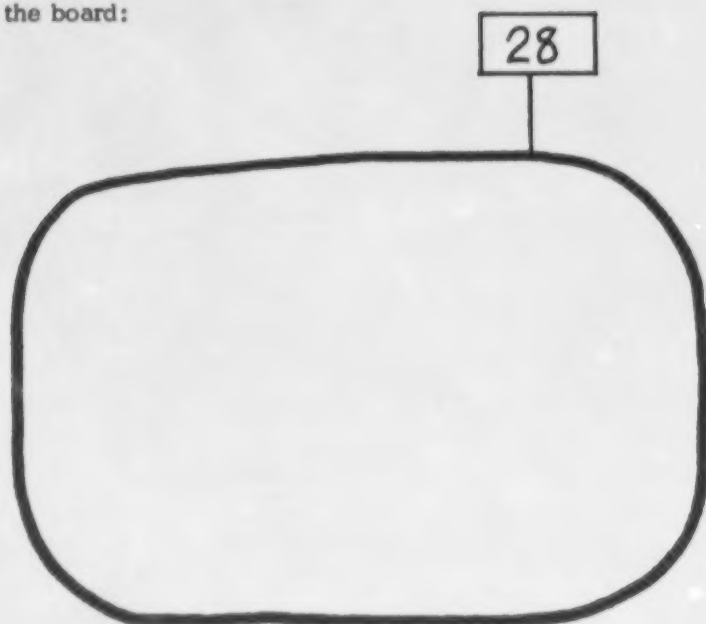
- Section 2.1:** Review decimal numbers, building on experiences from Part II of the CSMP Upper Primary Grades Program.
- Section 2.2:** Students represent various amounts of money on the Minicomputer. Using monetary situations as motivation, perform calculations with decimal numbers such as " 10×0.10 "; " 5×0.05 "; " $10.00 - 6.67$." Locate positions for various decimal numbers on the number line. (Worksheets N01, N02)
- Section 2.3:** Continue to develop situations similar to those of Section 2.2. Practice reading various Minicomputer configurations of decimal numbers. Explore multiplication by 10 on the Minicomputer. Build roads using decimal numbers.

3M

2.1 Halving and Decimal Numbers

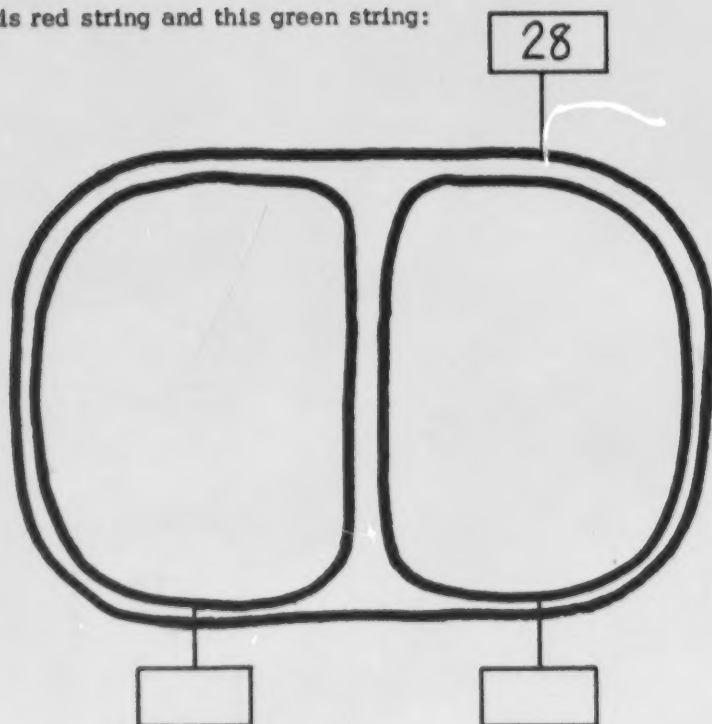
Exercise 1:

Draw on the board:



T: Let's pretend that there are 28 marbles inside the black string. I shall not draw them.

Draw this red string and this green string:



T: Half of the marbles are inside the red string and half of the marbles are inside the green string.

How many marbles are there inside each of the strings? (Answer: 14)

How do you know? (Answer: $14 + 14 = 28$ or $2 \times 14 = 28$ or $\frac{1}{2} \times 28 = 14$ or $28 \div 2 = 14$)

Write "14" in the red tag and in the green tag and write on the board:

$$\frac{1}{2} \times 28 = 14$$

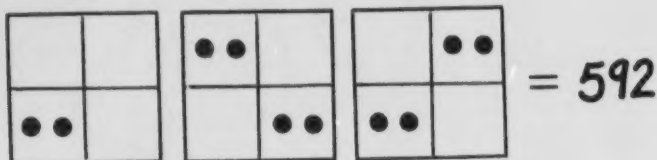
$$28 \div 2 = 14$$

Erase "28" and "14" and do a similar exercise with some other numbers.

For instance, 30; 48; 70; 100; 110; 250; 300; 314; 1,000; 1,050.

Exercise 2:

Slowly and progressively put all these checkers on the Minicomputer and have the students calculate this number:



T: Just by looking at the Minicomputer, can you immediately see what is $\frac{1}{2}$ of 592 (or 592 divided by 2)?

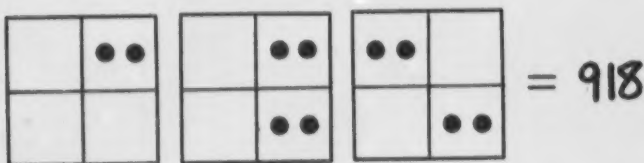
Have a student explain that the Minicomputer clearly shows that

$$592 = 2 \times 296.$$

Therefore,

$$\frac{1}{2} \times 592 = 296 = 592 \div 2.$$

Do a similar exercise with

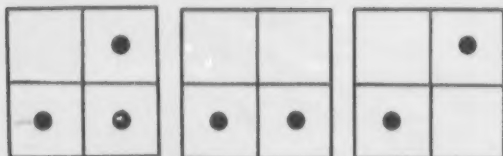


$$918 = 2 \times 459$$

$$\frac{1}{2} \times 918 = 459$$

Exercise 3:

Put on the board:



T: What is this number?

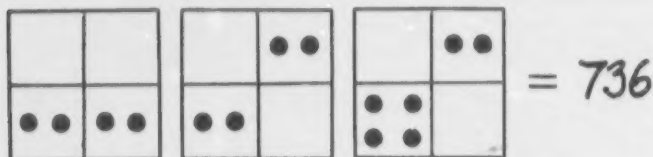
S: 736

T: Can you easily calculate " $\frac{1}{2}$ of 736"?
 Is the answer smaller or larger than 200?
 smaller or larger than 300?
 smaller or larger than 400?

Write on the board some of the students' guesses.

T: How can you calculate " $\frac{1}{2}$ of 736" with the help of the Minicomputer?

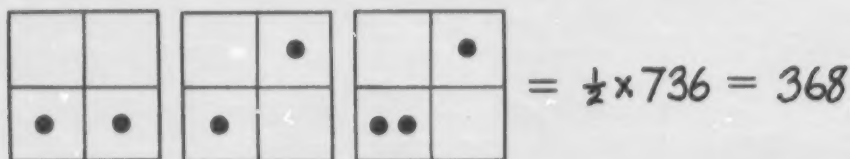
Let students make backward moves on the Minicomputer in order to pair the checkers. They will probably get:



T: What is $\frac{1}{2}$ of 736?

3M

Have a student remove half of the checkers and conclude:



Exercise 4:

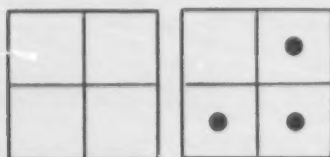
T: I have \$7 and I would like to share them fairly between my two friends. How much must I give to each of them?

Have the students write the answer (\$3.50) on a sheet of paper.

T: How could you give \$3.50 to a person?
What bills and what coins could you use?

Have the students suggest many ways of solving this problem.

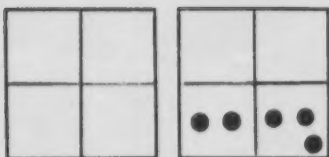
Put 7 on the Minicomputer:



T: I have put \$7 on the Minicomputer.
How can we use the Minicomputer to calculate " $\frac{1}{2}$ of 7"?

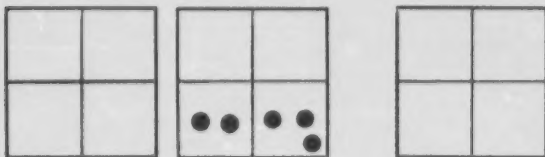
The students will probably suggest playing backward until they get the checkers in groups of two. Your students who did not study the CSMP curriculum in the third grade^{*} will probably be stuck after the first two moves.

^{*}Even if you used CSMP Mathematics for the Intermediate Grades, Entry Module I during the first semester of the fourth grade, all of your students will most likely be stuck at this point.



T: What must we do next?

Let the students think about this new² problem. Encourage them to express their ideas. In most of the classes where the question has been asked, at least one student suggested adding a new board to the right. If this doesn't happen, don't worry. Suggest it yourself. Place the new board to the right of the ones' board . . . but leave more space than usual between these boards.



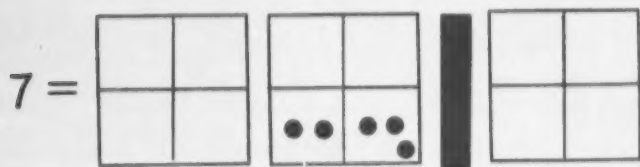
Perhaps a child will object:

S: But now it's not seven any more. It's 70.

(If nobody objects, you should raise the question of whether it's still seven.)

²Veteran CSMP students should have had some experience with this problem at the end of third grade. Some will probably know how to proceed, but all should need the exercise as review, at least.

T: That's right! In order not to forget that it's seven, let us put this green bar² between the boards:



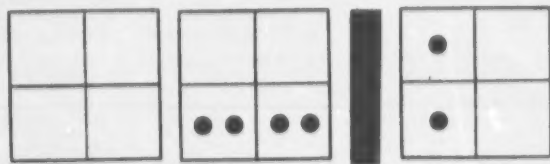
Emphasize that the green bar is there to remind us that this board (point) is still the ones' board.

T: What shall we do next?

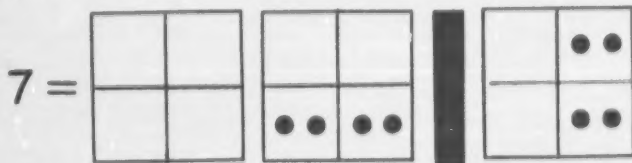
Usually, at this point, the children are confused and they often suggest some wrong plays. Don't worry about that. It's natural that the green bar disturbs them for a short while. Mistakes play an important role in every learning experience.

If this happens, play the game yourself and say simply:

T: One checker on this white square (point) is just the same as one checker on this brown square (point) and one checker on this red square (point).



Generally the students make the next two moves without any trouble.



²The bar should be drawn with green chalk directly on your chalkboard. Do it for this lesson and all subsequent lessons involving the Minicomputer representation of decimals.

Now have a student take one-half of 7.

$$\frac{1}{2} \times 7 =$$

T: How do you suggest we write the answer under the Minicomputer boards?

Different reactions are possible:

$$\frac{1}{2} \times 7 =$$

3 ■ 5
or 3 . 5
or 3 . 5

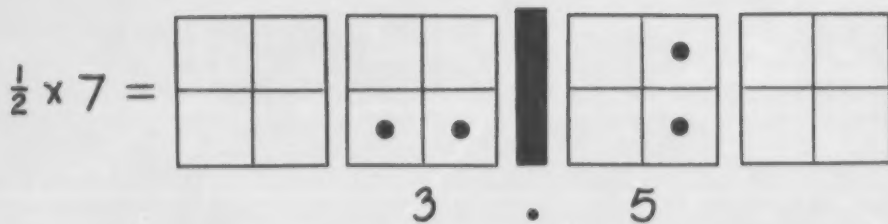
T: Very good! Which method of writing do you like best?

Let the children express their ideas. They will probably agree that the last method of writing is the easiest one because we don't always carry green pencils around with us . . . and also because this method of writing is in common use in stores.

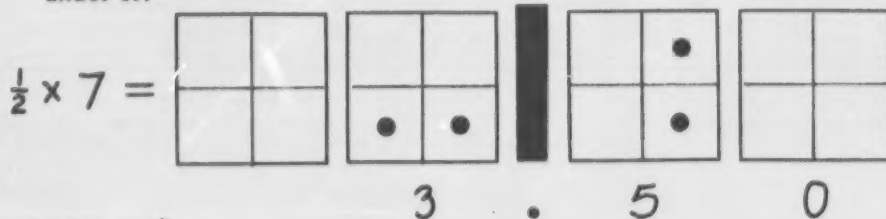
T: Is it written exactly the same way as in the stores?

S: No! They write "3.50."

Add a new board to the right without saying anything. Wait for the reactions of the class.



S: There are no checkers on the new board! We could write "0" under it.



Emphasize that*

$$3.5 = 3.50$$

and conclude for the second time that

$$\frac{1}{2} \times 7 = 3.50.$$

*Read "3.5" as "three point five." Read "3.50" as "three point five zero."

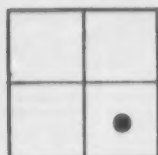
Exercise 5:

Write these three series of problems on the board and have the students solve as many of them as they can for the rest of the lesson:

★	★★	★★★
$\frac{1}{2} \times 24 =$	$\frac{1}{2} \times 434 =$	$\frac{1}{2} \times 570 =$
$\frac{1}{2} \times 40 =$	$\frac{1}{2} \times 870 =$	$\frac{1}{2} \times 990 =$
$\frac{1}{2} \times 46 =$	$\frac{1}{2} \times 1,080 =$	$\frac{1}{2} \times 37 =$
$\frac{1}{2} \times 30 =$	$\frac{1}{2} \times 3 =$	$\frac{1}{2} \times 1.50 =$
$\frac{1}{2} \times 34 =$	$\frac{1}{2} \times 25 =$	
$\frac{1}{2} \times 400 =$		

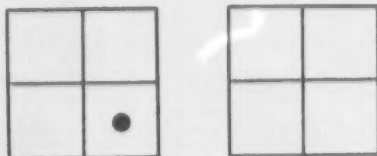
2.2 Dollars, Dimes and PenniesExercise 1:

Display one Minicomputer board and put one checker on the 1 - square.



T: This checker is for \$1. Could you show one dime on the Minicomputer?

Probably some children will remember the preceding lesson and will suggest that you put a new board on the right like this:



T: What can we do to remind us that this checker is for \$1?

The students will probably suggest that you draw the decimal point (or the green decimal line).

Draw the green line on the board and take the checker off.



T: Now put one dime on the Minicomputer.

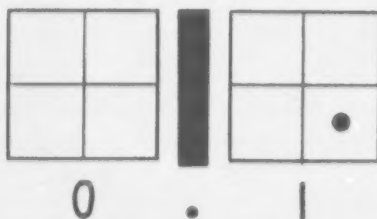
Have a student do it:



T: How do you write this number?

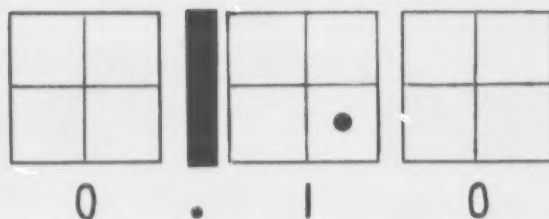
S: 0.1

Write "0.1" under the boards:



Add a new board to the right and say:

T: When I add a new board to the right, nothing is changed. It's still one dime but we can write it this way:

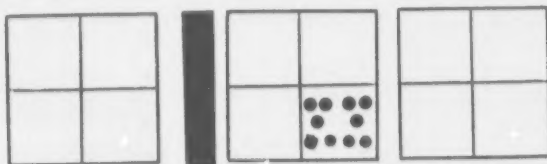


Write on the board:

$$0.1 = 0.10$$

T: Put 10 dimes on the Minicomputer.

Have a student put 10 checkers on the "dimes" - square:



Next, have students show that 10 dimes equal \$1 by making forward moves on the Minicomputer.

Write on the board:

$$10 \times 0.10 = 1$$

T: Can you put one penny on the Minicomputer?

Have a student do it:



T: How do we write this number?

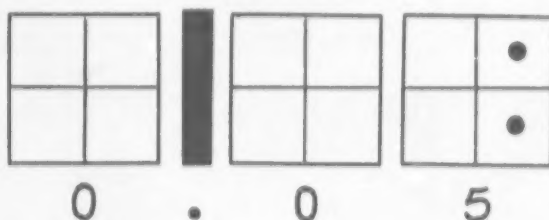
Have a student write "0.01" under the Minicomputer boards.

Let a student put 10 pennies on the Minicomputer, then ask several students to show that 10 pennies equal 1 dime by making forward moves.

Write on the board:

$$10 \times 0.01 = 0.10$$

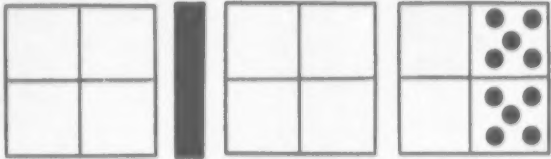
Ask a student to put one nickel on the Minicomputer:

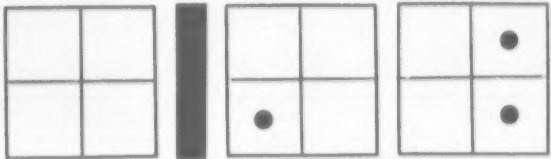


Have another student put five nickels on the Minicomputer:



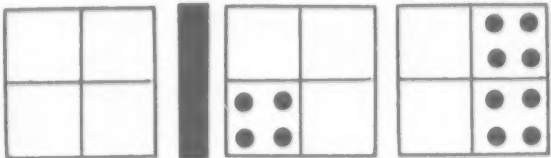
Ask several other students to show that five nickels equal one quarter by making forward moves.

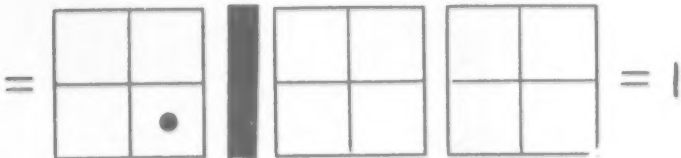
$$5 \times 0.05 =$$


$$=$$


0
.
2
5

Ask a student to put four quarters on the Minicomputer and have him show that four quarters equal \$1 by making forward moves.

$$4 \times 0.25 =$$


$$=$$


Exercise 2:

T: Let's pretend that you have one \$5 bill and that you have spent \$2.85 in a shop. How much money is left? Write your answer on a sheet of paper.

On the board write:

You have \$5.00
You spend \$2.85
How much money is left?

Have a student explain why \$2.15 is left.

Ask a similar question for:

You have \$10.00
You spend \$6.67
How much money is left?

Have a student explain why \$3.33 is left.

[illegible]

T: I would like to trade \$1 bill for two coins. How can I do it fairly?

S: Two 50¢ coins.

Have the students complete the first row of the chart. [Answer: "2" in the outside column, "2" in the Half-dollar column and nothing (or "0") in all the other columns.]

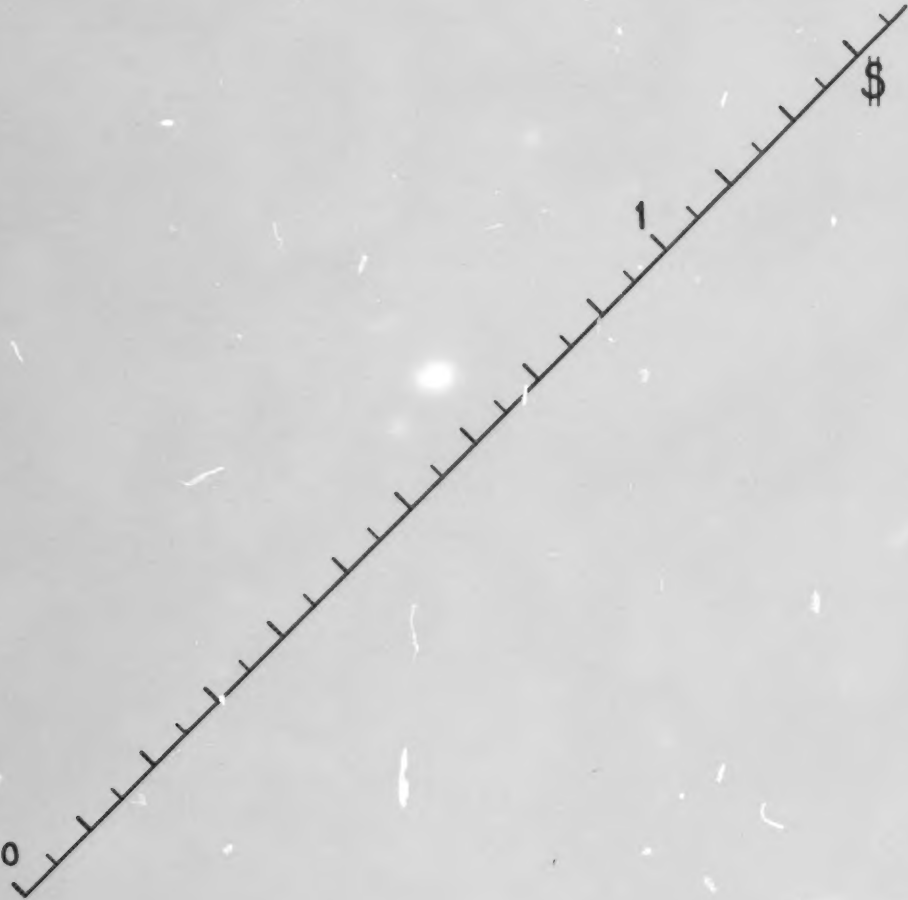
Ask a similar question successively for 3, 4, 5, 6, 10, 12, 15 and 20 coins. Have the students progressively complete the chart and share their solutions. When it's over, your board should look more or less like this:

Number of coins	Half-dollar .50	Quarter .25	Dime .10	Nickel .05	Penny .01
2	2				
3	1	2			
4		4			
5	1	1	2	1	
6	1 1	1	5 1	3	
10			10		
12		2	8	4 10	
15		3	1	1	10
20				20	

This chart doesn't give you an exhaustive list of the possible solutions but only the solutions that were given by the students of one of our fourth grade development classes.

Exercise 3:

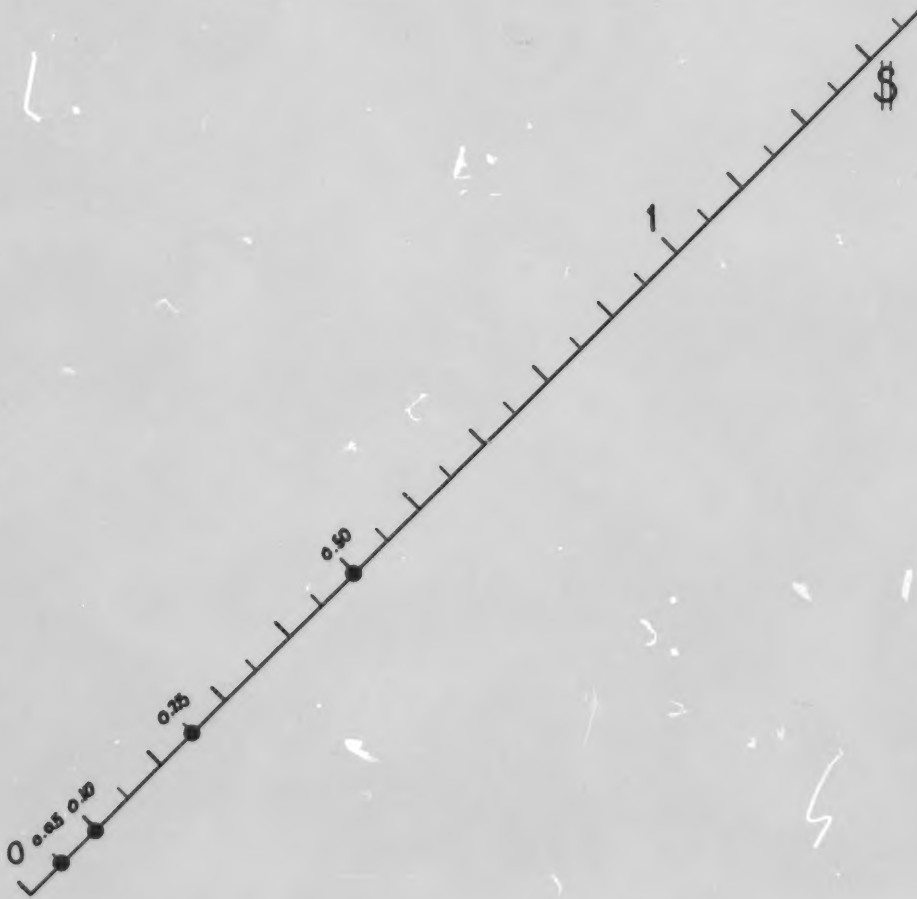
Give worksheet N02 to each student and draw the \$ number line on the board:



T: Can you put a dot for one dime on the number line?

Let a student show the dot on the board and have him label the dot ("0.10").
Have all the students label this dot on their worksheets.

Ask a similar question for one 50¢ coin, one nickel and one quarter.



8M

T: Where is the dot for one penny?

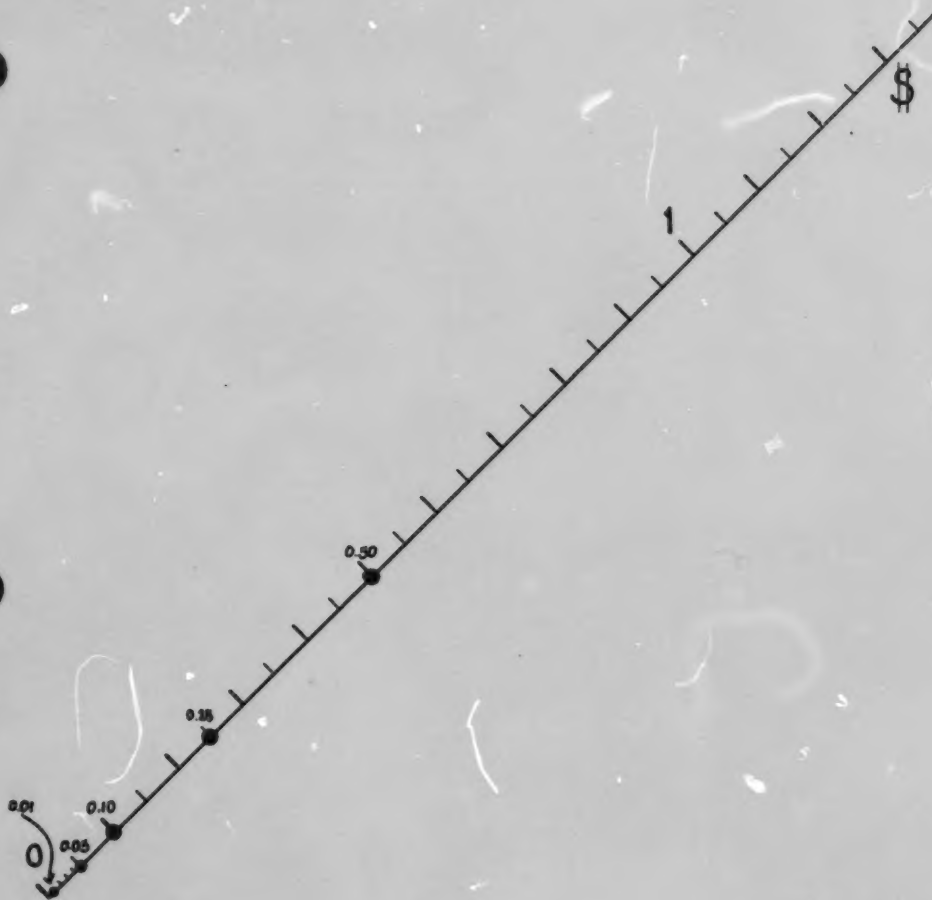
The students will probably show where this dot has to be, more or less.

T: We almost need a magnifying glass to draw it.

On another board (if you have one!), draw this part of the number line with a larger scale and put on the dot for one penny:

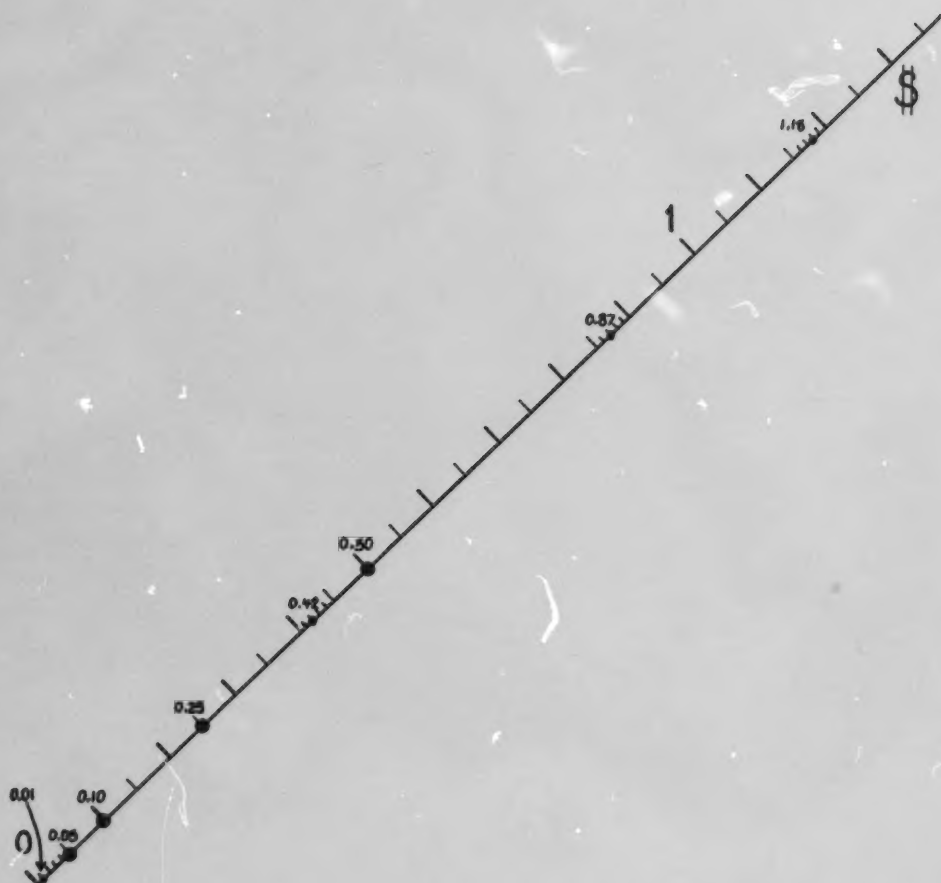


Then put the dot for one penny on the other picture:



Have the students put on dots successively for \$0.87; \$0.42 and \$1.18.

8M

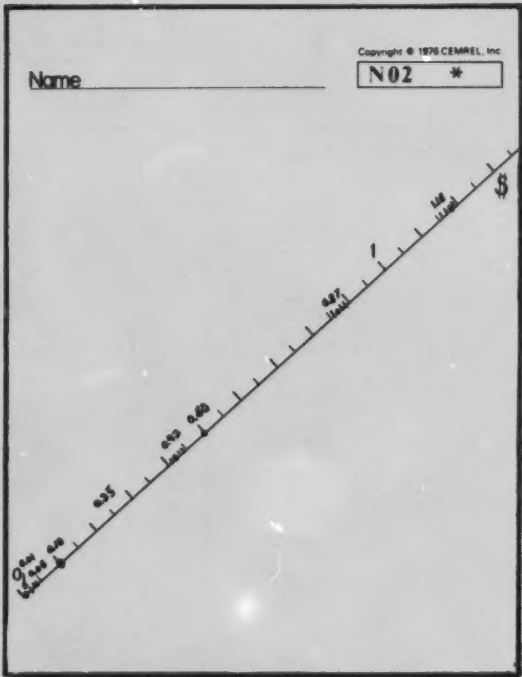


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Name _____

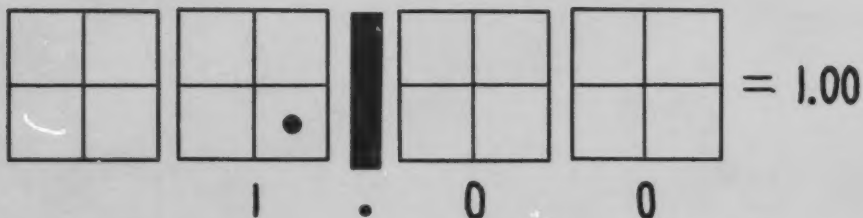
N01 *

Number of coins	Half-dollar \$.50	Quarter \$.25	Dime \$.10	Nickel \$.05	Penny \$.01
2	2				
3	1	2			
4		4			
5	1	1	2	1	
6	1	1	5	3	
10			10		
			8	4	
12		2		10	
15		3	1	1	10
20				20	



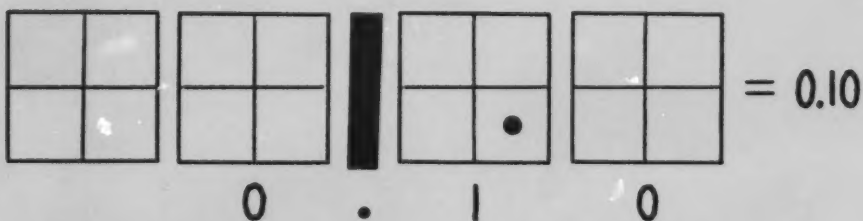
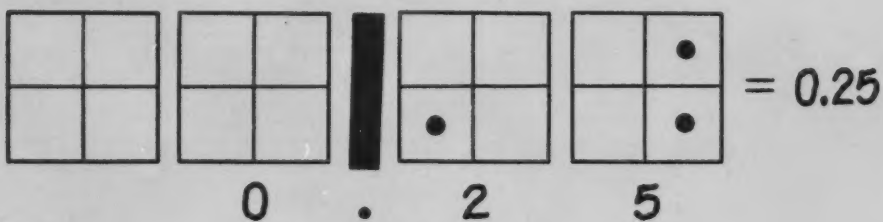
2.3 Assorted ProblemsExercise 1:

Put on the board:

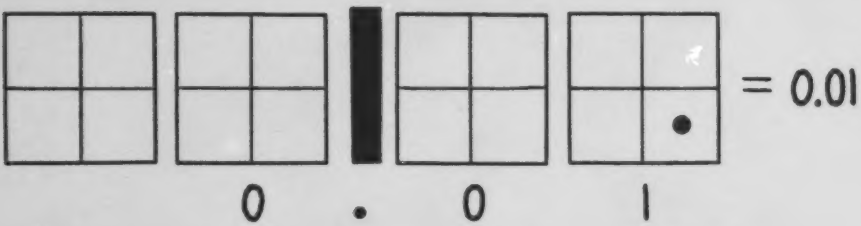
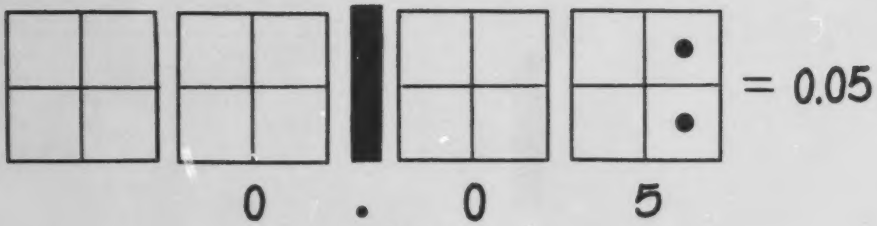


T: I have put \$1 on the Minicomputer.

Ask students to put on successively one quarter, one dime, one nickel and one penny:

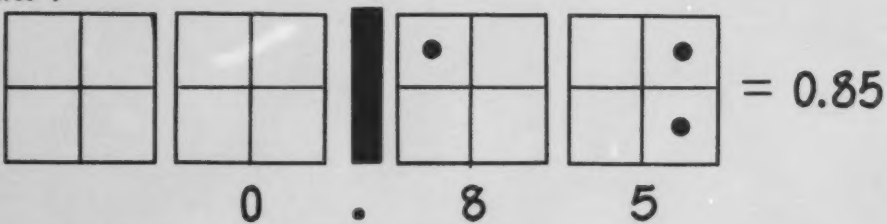


15M



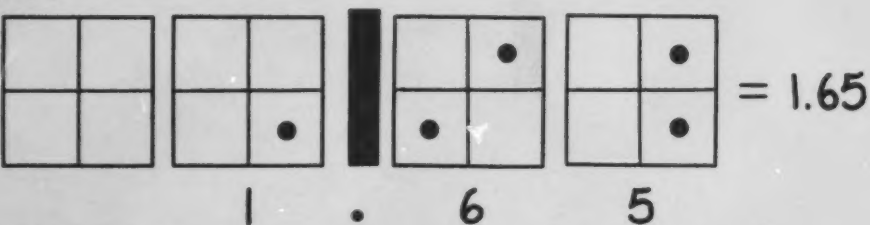
T: Put 3 quarters + 2 nickels on the Minicomputer.

Answer^{*}:



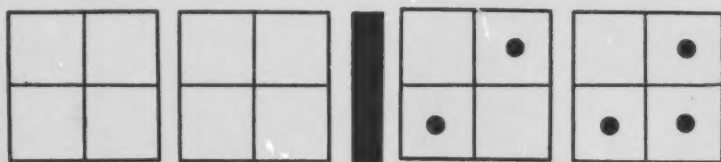
T: Put 6 quarters + 3 nickels on the Minicomputer.

Answer^{*}:



* If the students do not do the calculations in their heads before putting these numbers on the Minicomputer, make the necessary robot moves in order to get the standard configurations.

Put on the board:



T: How much money is there on the Minicomputer? Write the answer on your sheet of paper. (Answer: \$0.67)
How much must we add to get \$1?

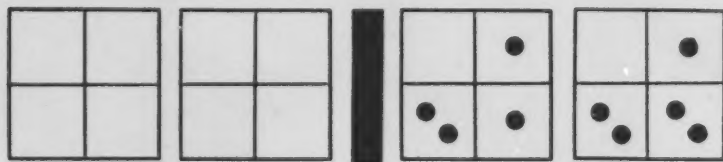
Write on the board:

$$\begin{array}{r} 0.67 \\ + \\ \hline 1.00 \end{array}$$

Ask a student to complete this calculation:

$$\begin{array}{r} 0.67 \\ + 0.33 \\ \hline 1.00 \end{array}$$

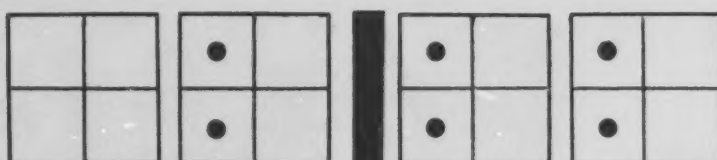
Let another student add 0.33 to the Minicomputer:



T: By making robot moves, let's make sure that the number on the Minicomputer is really 1 (or \$1.00).

Let several students take turns in making the robot moves.

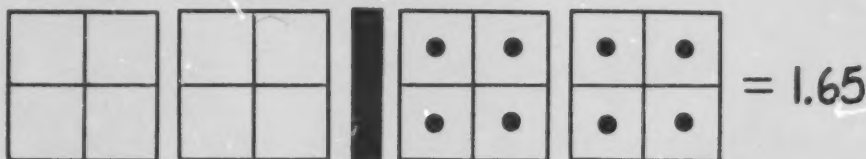
Put on the board:



T: What number is on the Minicomputer? Write it on your sheet of paper. (Answer: 11.10)

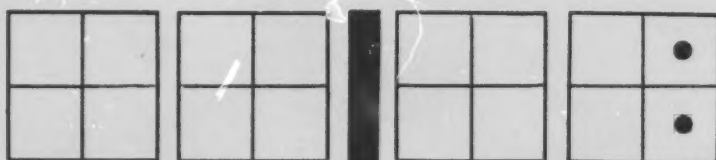
Have a student make robot moves to show that this number is 11.10.

Do a similar exercise with:



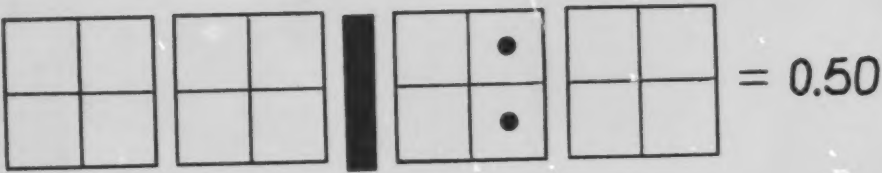
Exercise 2:

Put on the board:

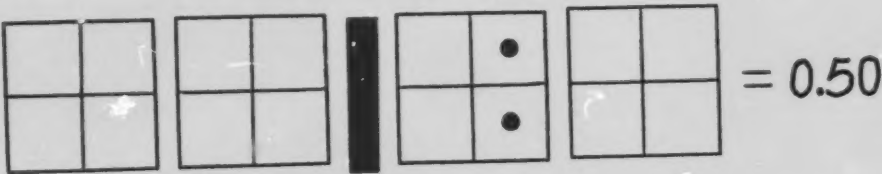


T: What number is on the Minicomputer? (Answer: 0.05)
Show 10×0.05 .

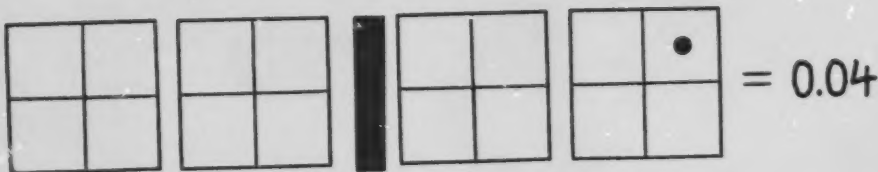
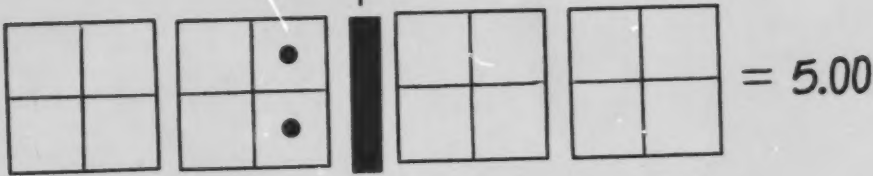
Answer:



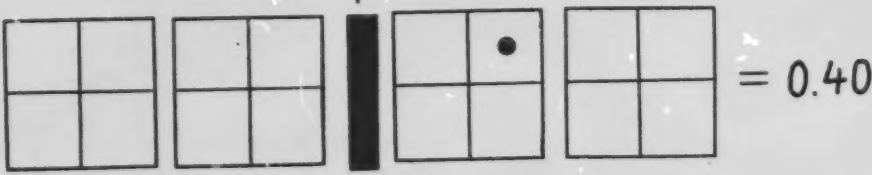
Proceed similarly with the following numbers:



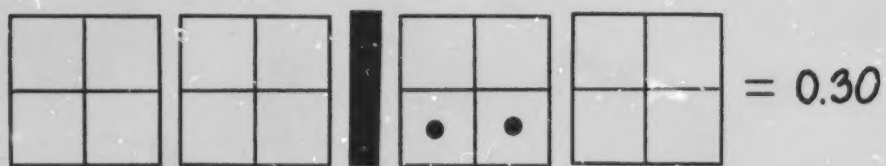
↓ 10x



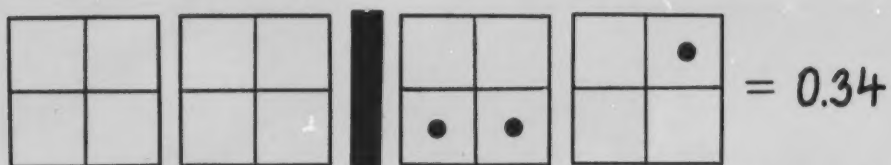
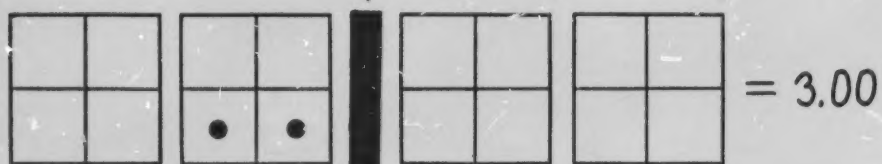
↓ 10x



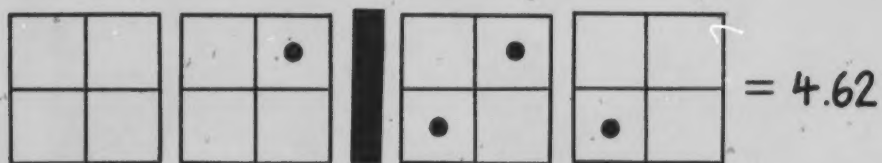
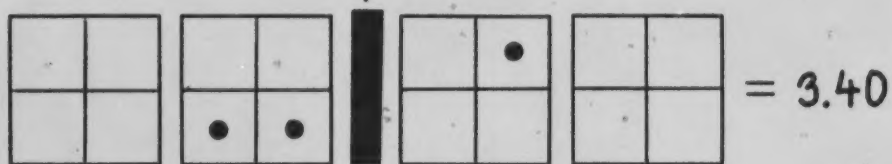
15M



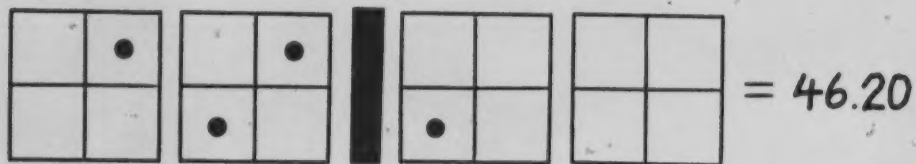
↓ 10 x



↓ 10 x



↓ 10 x



83

Write these calculations on the board and ask the students to complete them on a sheet of paper:

$$10 \times 0.05 =$$

$$10 \times 0.50 =$$

$$10 \times 0.55 =$$

$$10 \times 5.05 =$$

$$10 \times 5.50 =$$

$$10 \times 5.55 =$$

After a while, have the students complete the calculations on the board:

$$10 \times 0.05 = 0.50$$

$$10 \times 0.50 = 5.00$$

$$10 \times 0.55 = 5.50$$

$$10 \times 5.05 = 50.50$$

$$10 \times 5.50 = 55.00$$

$$10 \times 5.55 = 55.50$$

Emphasize the fact that

$$5 = 5.0 = 5.00$$

$$0.5 = 0.50$$

$$5.5 = 5.50$$

Ask the students to complete these calculations:

$$2 \times 0.80 =$$

$$3 \times 0.60 =$$

$$4 \times 0.70 =$$

Answers:

$$2 \times 0.80 = 1.60$$

$$3 \times 0.60 = 1.80$$

$$4 \times 0.70 = 2.80$$

Exercise 3:

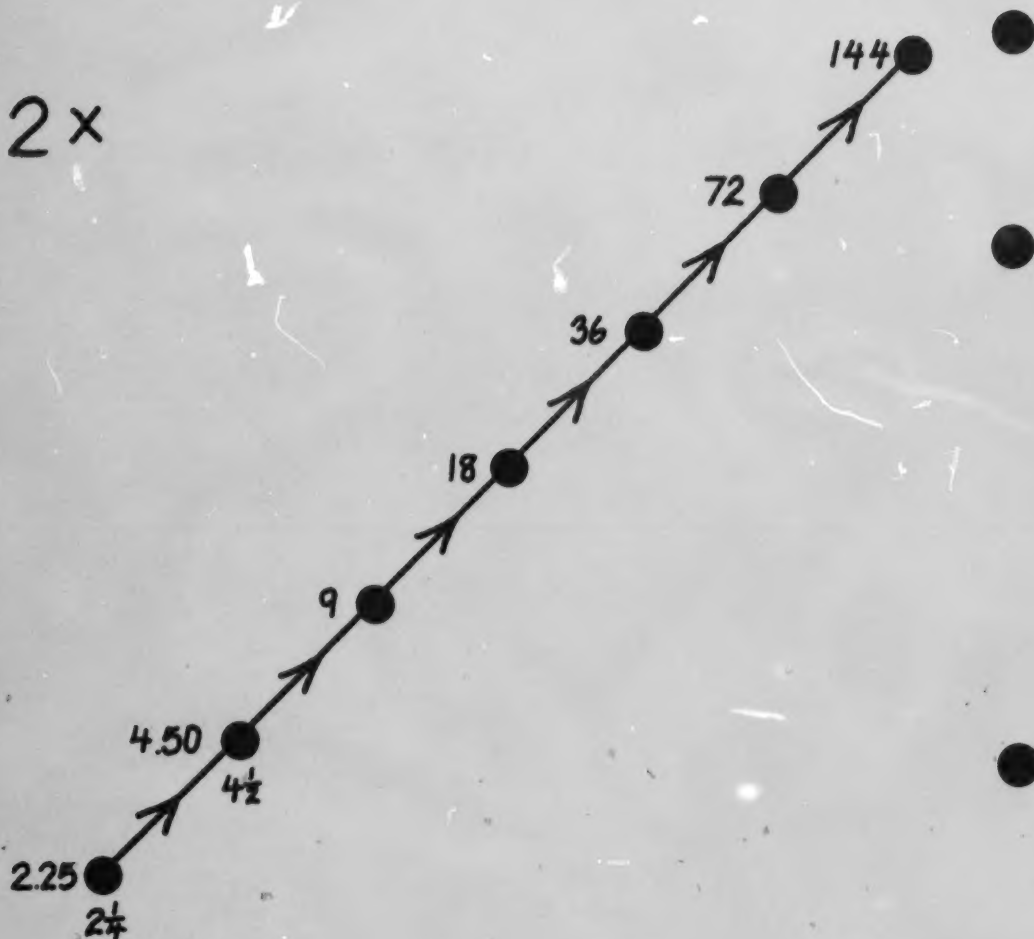
Draw on the board:

 $2 \times$ 

Ask the students to copy this picture and to label all the dots. After a while, let several students take turns to complete the picture:

15M

2x



Write on the board:

Problem *

Problem **

2x
-1

100 ●

2x
-0.10

10 ●

7 ●

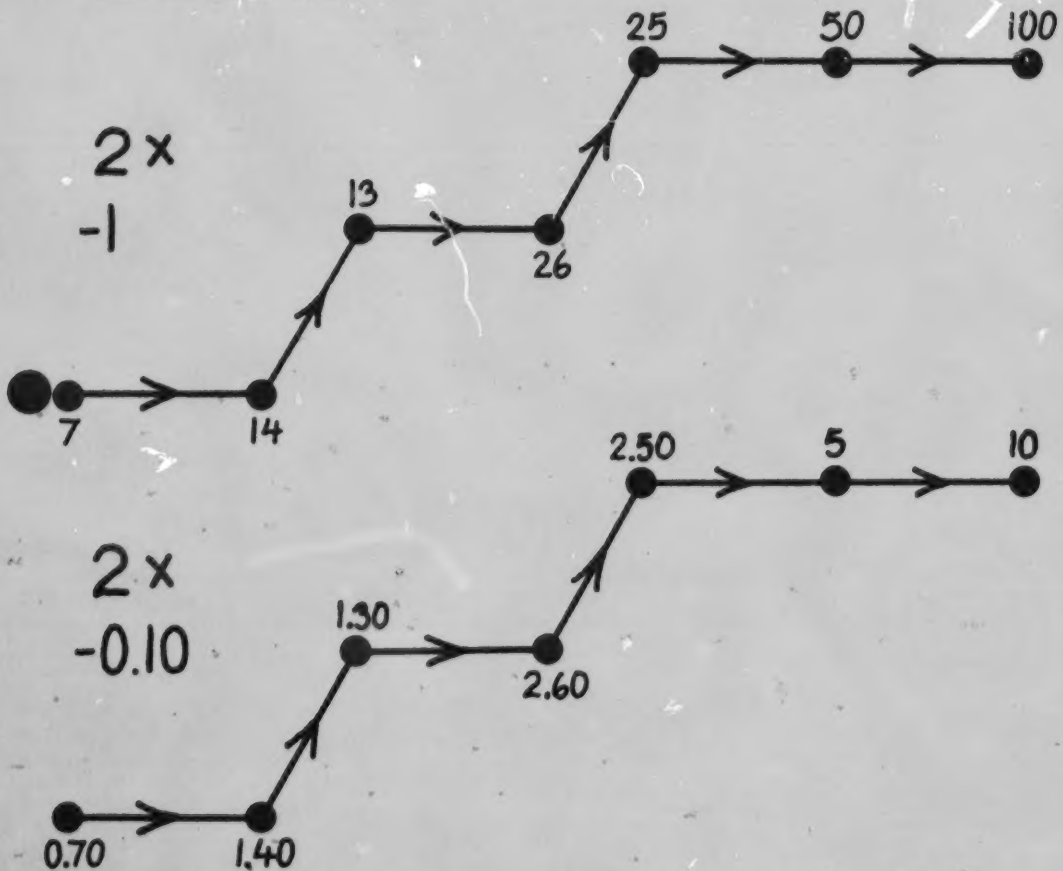
0.70 ●

86

112

T: First build a red-blue road from 7 to 100. Afterwards build a red-blue road from 0.70 to 10. In each problem, try to use the smallest number of arrows.

Let the students work on their own for a while. As soon as one student has built a road with arrows, tell the other students that it's possible to solve each problem with only six arrows. Before the end of the lesson, ask a student who has solved the problem with six arrows to draw his or her solution on the board:



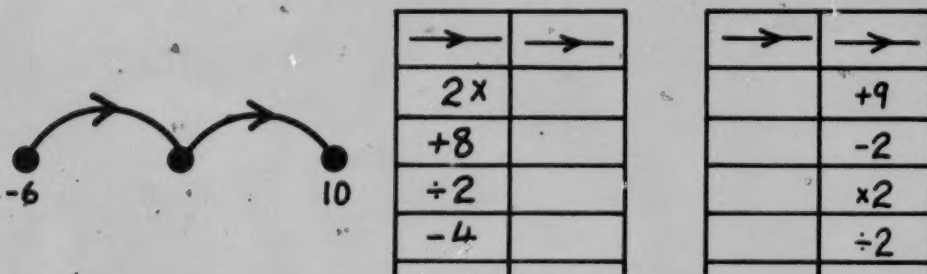
Some students should notice that the same arrow picture can be used to solve both problems.

NEGATIVE INTEGERS

Work with negative integers is introduced in the CSMP First Grade Program. By the end of Part II of the CSMP Upper Primary Grades Program (third grade) students have had extensive exposure to various interpretations of the integers and some experience with the addition operation on this set.

In this chapter the Minicomputer and arrow language activities of Chapter 1 are extended to the realm of the integers. Order among the integers is stressed in several contexts. The Minicomputer and the language of arrows are used to pose questions about the representation of positive and negative integers. One such situation, using arrow diagrams, is the following:

The students are given the following arrow diagram and table of numerical functions and invited to complete the tables, identifying the red or blue arrow, as the case may be, and determining what number is represented by the unlabeled dot in each case.



This type of situation offers rich opportunities for honing numerical skills as well as a natural environment for argumentation (i.e., If the red arrow is for $2\times$ then what could the blue arrow be for?)

NOTES ON NOTATION:

- 1) Common mathematical notation is replete with ambiguities. For example, there is usually no distinction made on the printed page

between the function "subtract 3" and the number "negative 3," both being denoted by "- 3." It is left to context to determine which of these two objects is intended when one sees "- 3" written in a piece of mathematics. CSMP feels that it is exceedingly detrimental to the learning process to introduce the blurring effect that necessarily attends such an ambiguity at precisely the time when the children are trying to clarify in their minds exactly what functions and numbers are. Hence CSMP makes a distinction in notation between the function and the number, leaving it until fifth or sixth grade before children are expected to be able to deal with multiple uses of symbols. The convention is as follows:

A numeral preceded by a minus sign in usual position always represents a function. Thus, for example, "- 14" denotes the function "subtract 14."

A negative number is denoted in either of two possible ways:

- i) a numeral preceded by a raised minus; thus " $\overline{-}14$ " denotes the number "negative 14."
- ii) a numeral surmounted by a "hat"; thus " $\hat{14}$ " also denotes the number "negative 14."

The double notation for negative numbers is an attempt to instill a certain spirit of flexibility in the children while at the same time avoiding any confusion with other areas. This flexibility will stand the children in good stead when the time does eventually come for them to start using the common, ambiguous notation. In the same spirit, two types of negative checkers have been used in Minicomputer activities in previous parts of the CSMP curriculum: on some occasions, positive and negative numbers were represented on the Minicomputer by red and blue checkers, respectively; on other occasions, checkers with "hats" on them ($\hat{}$) were used as negative checkers.*

*In the Teacher's Guide for this semester's work we have only used the latter type of checkers, but your students may suggest the former method (using any two colors, not necessarily red and blue). You may use both types for the sake of variety and flexibility if you wish.

As in all the areas of the curriculum, if a child meets some notation and seems temporarily to have forgotten what it signifies, a gentle, brief reminder will suffice to put him or her back on the track again. Above all, don't make a big issue out of it!

- 2) The terms "opposite" and "0 - friend" may appear from time to time in student responses. Both terms mean the same thing, namely, the additive inverse (of a number). Two numbers are additive inverses if their sum is 0. For example, 5 and $\hat{5}$ (or -5) are opposites, 0 - friends and additive inverses of each other since $5 + \hat{5} = 0$ (or $5 + -5 = 0$).

CAPSULE LESSON SUMMARIES

Section 3.1: Compare integers on the number line with respect to order. Use the Minicomputer to explore such questions as "Which numbers can be represented on the ones' board with exactly two positive and two negative checkers?"
(Worksheets N03, N04, N05)

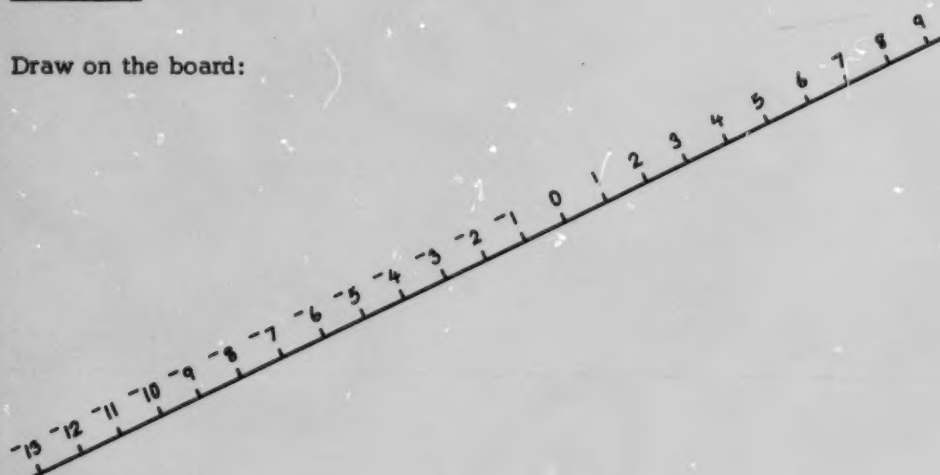
Section 3.2: Repeat the experiences of Chapter 1 in the context of the integers. Play Minicomputer Golf with negative checkers. Introduce a new type of problem: "Put -17 on the Minicomputer using just two checkers, positive or negative."

Section 3.3: Extend and develop the exercises of Section 3.2. Extend more of the situations from Chapter 1 to include the integers. Play Minicomputer Golf.

3.1 Negative Integers on the Number Line and on the Minicomputer

Exercise 1:

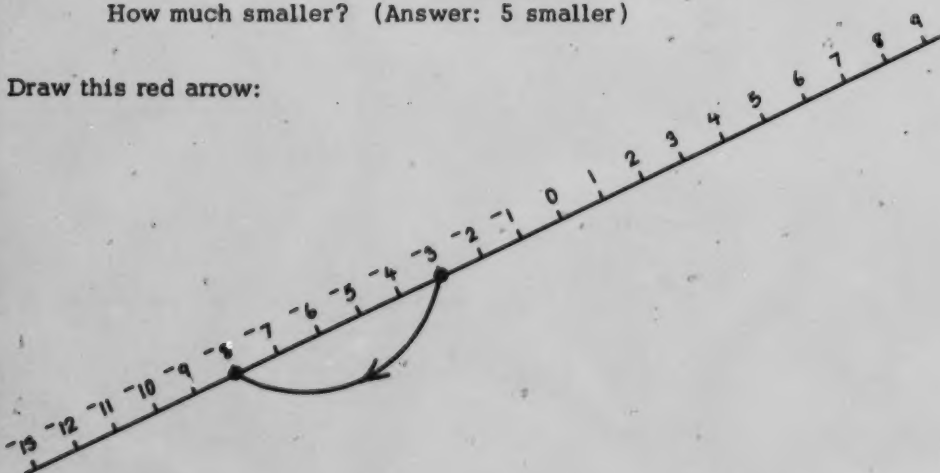
Draw on the board:



Point to "-3" and "-8" and ask:

- T: Which is the smaller number, -3 or -8? (Answer: -8)
 How much smaller? (Answer: 5 smaller)

Draw this red arrow:



T: What could this red arrow be for? (Answer: -5)

Write "-5" in red on the board.

Write on the board:

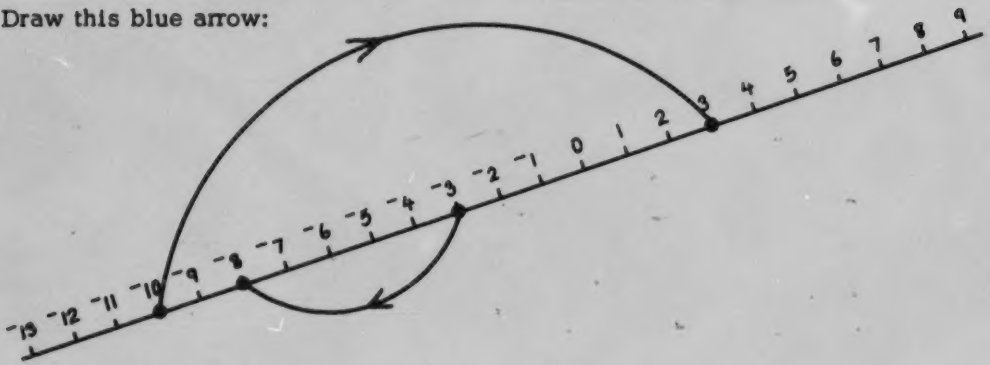
$$-3 - 5 = -8$$

Point to "-10" and "3" and ask:

T: Which is the larger number, -10 or 3? (Answer: 3)

How much larger? (Answer: 13 larger)

Draw this blue arrow:



T: What could this blue arrow be for? (Answer: +13)

Write "+13" in blue on the board.

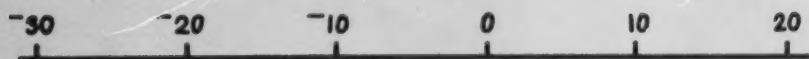
Write on the board:

$$-10 + 13 = 3$$

Do a similar exercise with two other pairs of numbers.

Exercise 2:

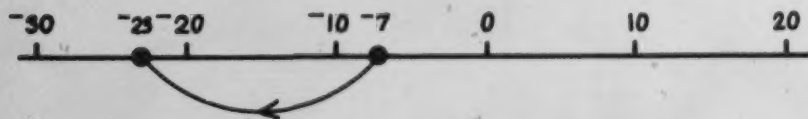
Draw on the board:



Have a student locate -7 and -23 on the number line and then ask:

T: Which is the smaller number, -7 or -23 ? (Answer: -23)
 How much smaller? (Answer: 16 smaller)

Draw this red arrow:



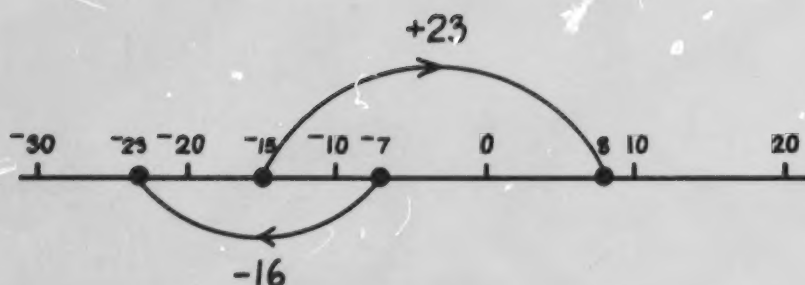
T: What could this red arrow be for? (Answer: -16)

Write " -16 " in red on the board.

Write on the board:

$$-7 - 16 = -23$$

Do a similar exercise with -15 and 8 and draw this blue arrow:



You can do similar exercises with one or two other pairs of numbers if you wish.

Exercise 3:

Ask a student to put two positive and two negative checkers on the ones' board of the Minicomputer (where he or she wishes) and have the other students calculate this number. For instance,

●	●
⊗ ⊗	

= 8

Have all the students do a similar exercise (with two positive and two negative checkers) on their desk Minicomputers and have some of them put their number on the board.^{*}

^{*} Throughout this text, when "on the board" is used in a Minicomputer exercise, it means "on the demonstration Minicomputer."

T: What is the largest number that you can get?

Answer:

$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline & \ominus \ominus \\ \hline \end{array} = 14$$

T: What is the smallest number that you can get?

Answer:

$$\begin{array}{|c|c|} \hline \ominus \ominus & \\ \hline & \bullet \bullet \\ \hline \end{array} = -14 = \overset{\wedge}{14}^*$$

Give worksheets N03, N04 to each student and ask them to find all the numbers that can be put on the ones' board with two positive and two negative checkers.

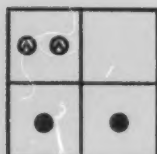
After a while, ask a student to put 13 on the board with two positive and two negative checkers.

$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \ominus & \ominus \\ \hline \end{array} = 13$$

* See the NOTE ON NOTATION in the introduction to this chapter for a discussion concerning this dual notation for negative numbers. Veteran CSMP students are more familiar with the " $\wedge$ " notation and may need to be reminded early in the semester that the raised minus sign is also used to denote negative numbers. A mixture of these two notations will occur throughout this guide as well as in the corresponding student materials without further comment.

T: Did you find -13 ?

S: It's easy. You just have to switch the positive and the negative checkers.


 $= -13$

Ask a similar question for 12 and -12 ; 11 and -11 ; 10 and -10 . Then let the students continue working on their own.

For the students who have found all the numbers from -14 to 14, worksheet N05 is available.

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Name _____

N03 *

Find all the numbers that can be put on the ones' board with two positive and two negative checkers. Write them on both sides of this worksheet.

 = 0	 = 1	 = 2
 = 3	 = 4	 = 5
 = 6	 = 7	 = 8
 = 9	 = 10	 = 11
 = 12	 = 13	 = 14

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Name _____

N04 *

 = -1	 = -2	 = -3
 = -4	 = -5	 = -6
 = -7	 = -8	 = -9
 = -10	 = -11	 = -12
 = -13	 = -14	

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Name _____

N05 **

Write these numbers with three checkers (positive or negative).

 = 17	 = 69
 = 37	 = 66
 = 71	 = 36
 = 73	 = 87
 = 24	 = 99

N03

(Many other solutions are possible.)

N04

(Many other solutions are possible.)

N05

(Other solutions are possible.)

3.2 Minicomputer Golf with Positive and Negative Checkers #1

Exercise 1:

Put numbers on the Minicomputer using both positive and negative checkers and have the students decode them. For instance,

	●

⊗	

 $= 8$

	⊗

●	

 $= -8$

●	

	⊗
	⊗

 $= 75$

⊗	

	●
	●

 $= \hat{75}$

7M

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline \odot & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \bullet \\ \hline & \\ \hline \end{array} = \overline{16}$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \odot \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline \bullet & \bullet \\ \hline \end{array} = \overline{37}$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \odot \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \odot \\ \hline & \bullet \\ \hline \end{array} = \hat{99}$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \odot \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} = \overline{998}$$

$$\begin{array}{|c|c|} \hline & \bullet \odot \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline \bullet & \odot \odot \\ \hline \end{array} \begin{array}{|c|c|} \hline & \odot \\ \hline \bullet \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \odot \\ \hline & \bullet \bullet \bullet \bullet \\ \hline \end{array} = 0$$

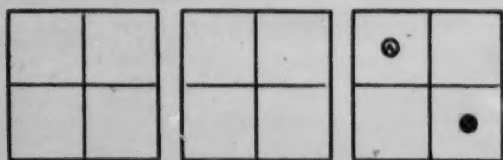
$$\begin{array}{|c|c|} \hline & \\ \hline & \bullet \\ \hline \end{array} \begin{array}{|c|c|} \hline \odot & \\ \hline \odot & \odot \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \\ \hline \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \odot \\ \hline & \bullet \\ \hline \end{array} = \hat{3}$$

100

125

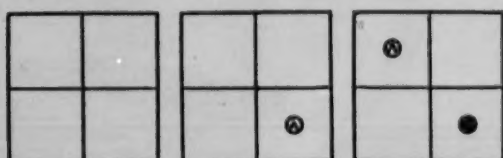
Exercise 2:

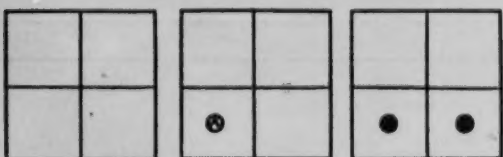
Ask the students to put -7 on their desk Minicomputers with two checkers (positive or negative) and have one student put the answer on the board:

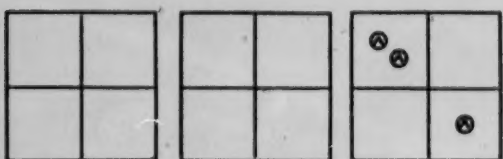

 $= -7$

T: Put -17 on with three checkers.

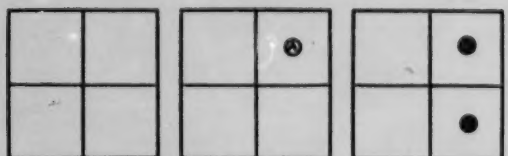
Several answers are possible.


 $= -17$

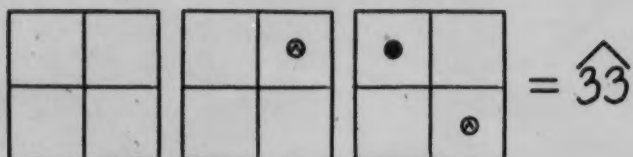

 $= -17$


 $= -17$

T: Put -35 on with three checkers.

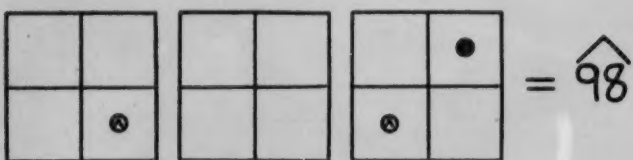
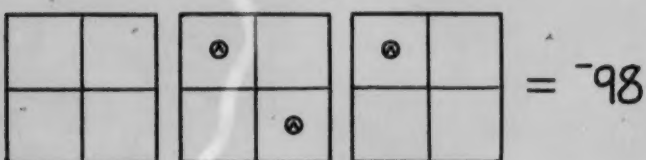
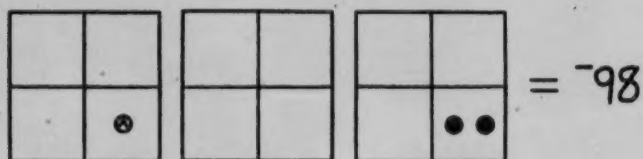

 $= -35$

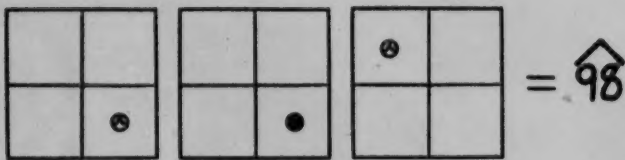
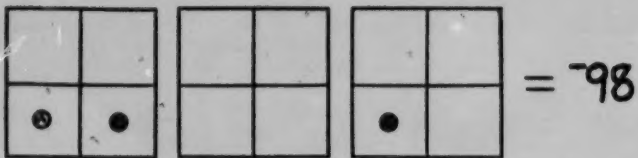
T: Put $\overline{33}$ on with three checkers.



T: Put $\overline{98}$ on with three checkers.

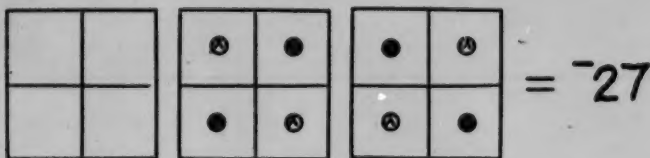
Encourage the students to find several solutions:



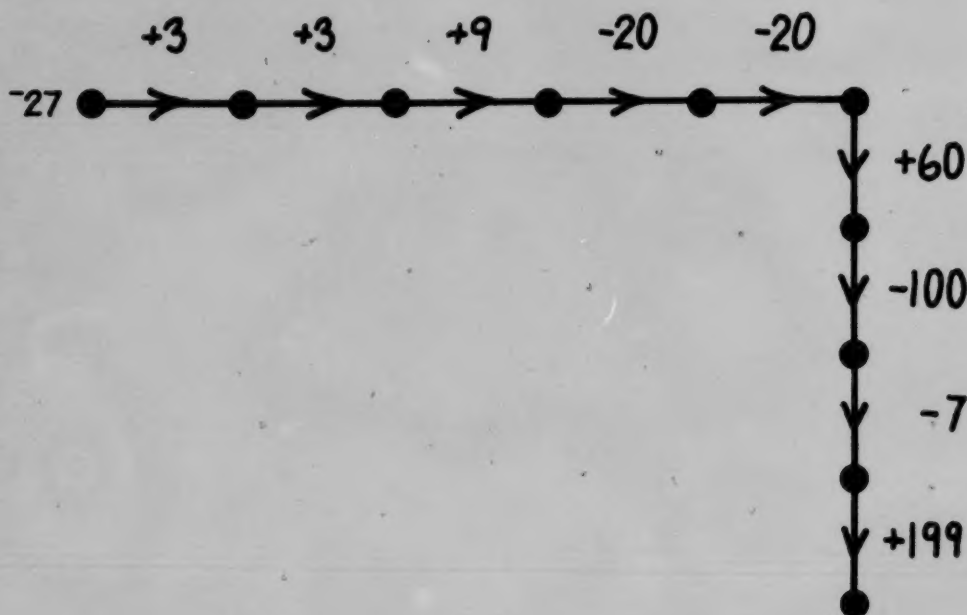


Exercise 3:

Slowly and progressively put all these checkers on the board and have the students calculate this number:

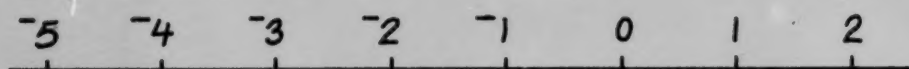


Draw this arrow picture on the board and ask the students to copy it on a sheet of paper:



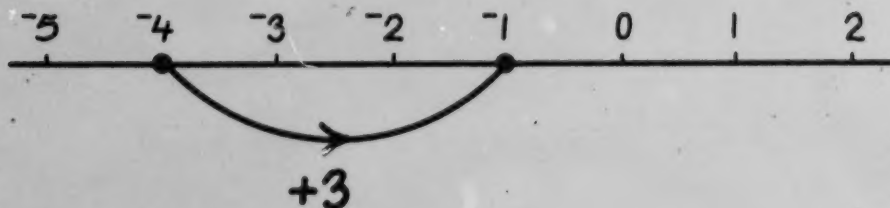
T: The first arrow starts from the number on the Minicomputer (-27) and asks you to make the number three bigger by moving exactly one checker from one square to another square. How can you do it? (Answer: By moving the positive checker from the 1 - square to the 4 - square or by moving the negative checker from the 4 - square to the 1 - square.)

If some students suggest the last move, you could draw this part of the number line on the board and ask:



- T: Which is the bigger number, -4 or -1 ? (Answer: -1) How much bigger is -1 than -4 ? (Answer: three bigger)

Draw a $+3$ arrow from -4 to -1 on the number line.



Then have a student make one of the two possible $+3$ moves on the Mini-computer.

- T: What number is on the Minicomputer now?

S: $-27 + 3 = -24$

Write " -24 " on the arrow picture.

- T: What do we have to do next?

S: Make the number three bigger again.

Have a student move a checker from -4 to -1 (or from 1 to 4, depending on which was done as the first move).

- T: What number is on the Minicomputer now?

S: $-24 + 3 = -21$

Write " -21 " on the arrow picture.

Follow similar steps for the other arrows:

+9

: Move a negative checker from the 10 - square to the 1 - square.
The new number is -12 .

-20

: Move a positive checker from the 40 - square to the 20 - square.
The new number is -32 .

-20

: Move a negative checker from the 80 - square to the 100 - square.
The new number is -52 .

NOTE: The preceding two moves can be interchanged.

+60

: Move a positive checker from the 20 - square to the 80 - square.
The new number is 8.

-100

: Move a negative checker from the 100 - square to the 200 - square.
The new number is -92 .

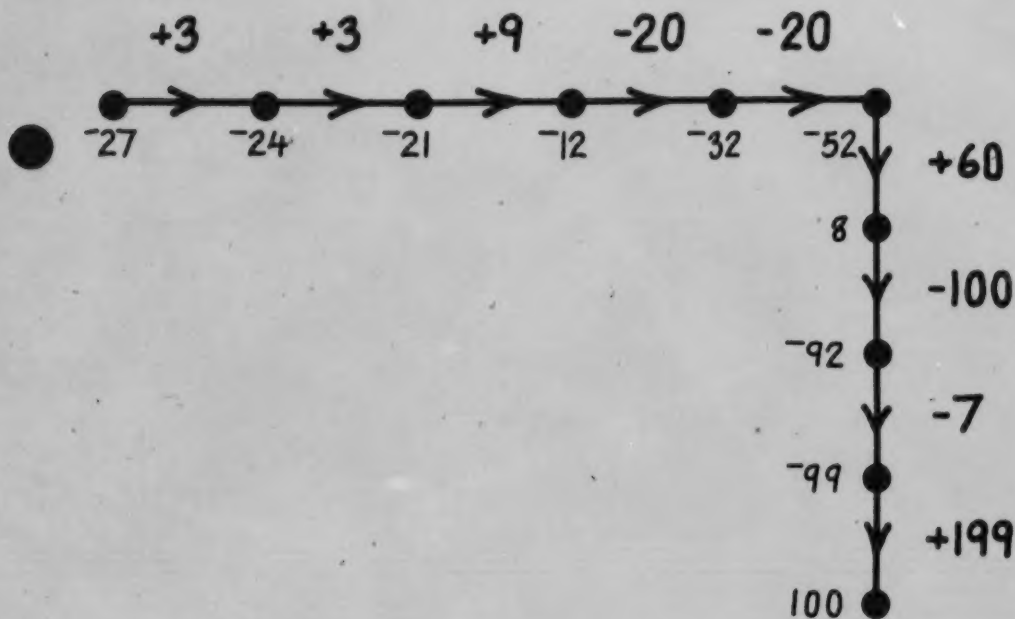
-7

: Move a positive checker from the 8 - square to the 1 - square.
The new number is -99 .

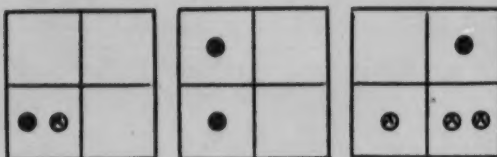
+199

: Move a positive checker from the 1 - square to the 200 - square.
The new number is 100.

The arrow picture has been progressively completed



and the result of moving the checkers on the Minicomputer is as follows:



T: What number should be on the Minicomputer now?

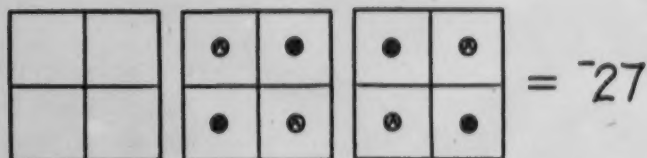
S: 100

Have the students calculate the number on the Minicomputer to see if it's correct.

Exercise 4:

T: Today we will play the Minicomputer Golf game with positive and negative checkers.

Put all these checkers on the Minicomputer and have the students calculate this number:



Divide your class in three groups. One group will be Team A, another group will be Team B and the third group will receive the hand-calculators^{*} and will have to check progressively the results of the game. (If you have no calculators, the third group could be Team C.)

Write on the board:

Goal: 100

Team A: →
Team B: →

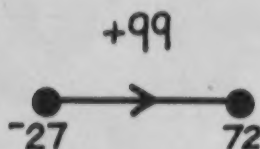
T: We are starting from -27 and we have to reach 100 by moving checkers from one square to another square. Who would like to play first for Team A?

Let's suppose that the first player of Team A moves a positive checker from the 1 - square to the 100 - square. We reproduce here the results of a game played in one of our fourth grade development classes.

^{*} The number of students in the third group should not exceed the number of available hand-calculators.

- T: Is the number on the Minicomputer bigger or smaller than -27 ?
 (Answer: Bigger)
 How much bigger? (Answer: 99 bigger)
 What number is on the Minicomputer now? (Answer: $-27 + 99 = 72$.
 This is to be answered by those having calculators.)

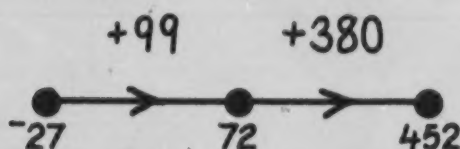
Draw this red arrow on the board:



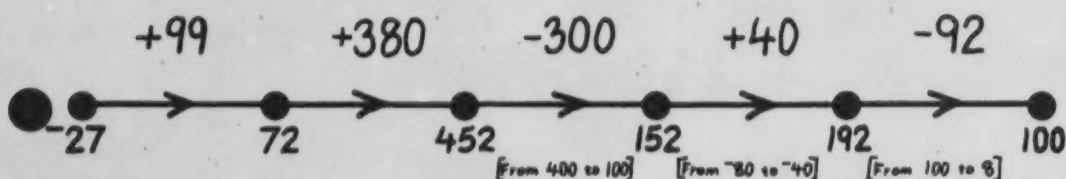
Now it's the turn of a player from Team B. He moves a positive checker from the 20 - square to the 400 - square.

- T: Is the number on the Minicomputer bigger or smaller than 72?
 (Answer: Bigger)
 How much bigger? (Answer: 380 bigger)
 What is the number on the Minicomputer now? (Answer:
 $72 + 380 = 452$)

Draw this blue arrow on the board:

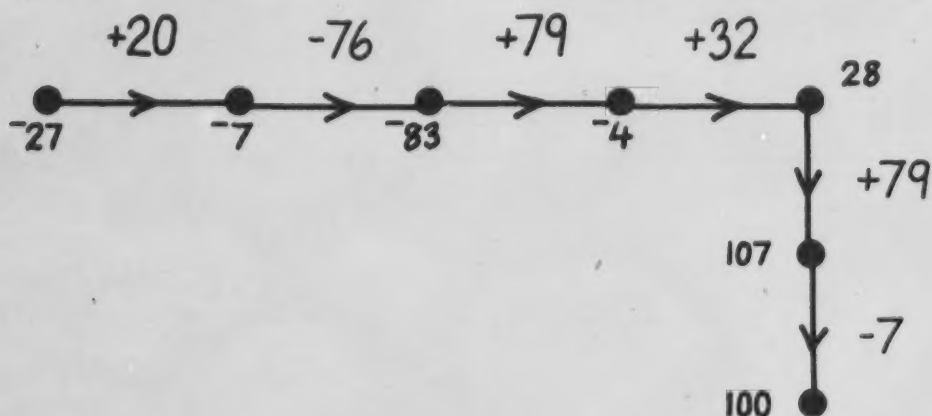
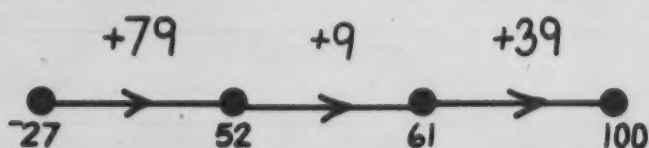
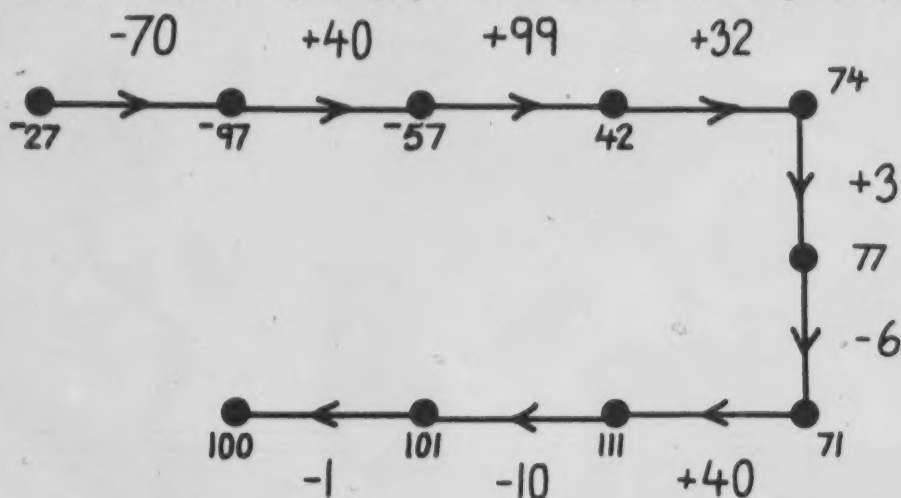


Follow similar steps for the next players, alternately from Team A and Team B. The results of the game are described by this red - blue road. Team A was the winner.



Exchange the roles between the three groups and have them play the same game again.

To show you how many possibilities any one game may have, we include the results of three other games played in our fourth grade development classes.



3.3 Minicomputer Golf with Positive and Negative Checkers #2

Exercise 1:

Ask the students to put 19 on their desk Minicomputer with exactly two checkers (positive or negative) and have one student put the answer on the board:

$$19 =$$

●	

	⊗

Ask a similar question for the following numbers:

(NOTE: For some of these problems, there is more than one answer.)

$$-32 =$$

	⊗

●	

$$\hat{76} =$$

⊗	

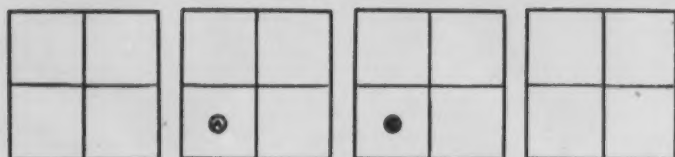
	●

$$180 =$$

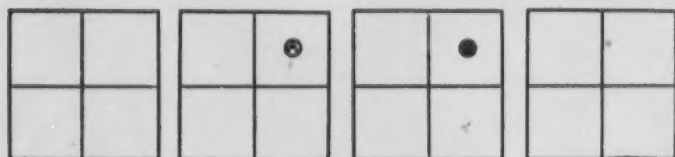
●	

⊗	

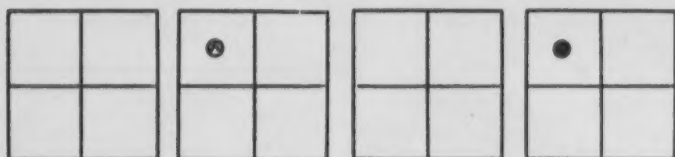
$$^{-}180 =$$



$$\textcircled{3}60 =$$



$$^{-}792 =$$



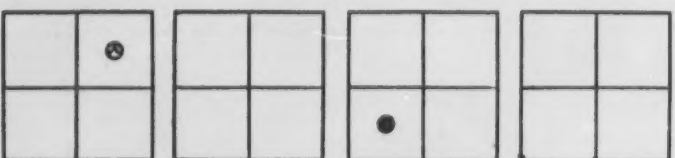
$$1,996 =$$



$$\textcircled{1},996 =$$

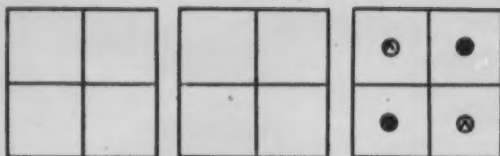


$$\textcircled{3},980 =$$



Exercise 2:

Put on the board:

T: What is this number? (Answer: -3)

Draw on the board:



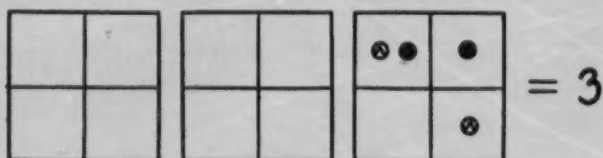
T: What could this red arrow be for?

S: $+6$.S: You are my 0 - friend^{*}.S: You are my opposite^{*}.Write " $+6$ " in red by the arrow.T: By moving one checker from one square to another square, can you transform -3 to 3?

S: Yes, by moving the checker from the 2 - square to the 8 - square.

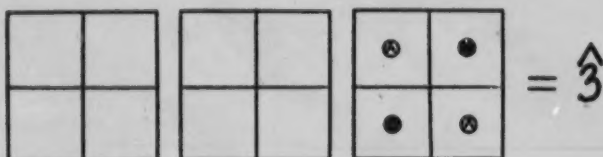
^{*} See NOTE ON NOTATION in the Introduction to this chapter.

Have the students check to see if the number now on the Minicomputer is really three.

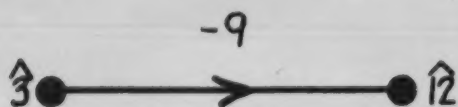


Follow similar steps for exercises 3 to 5.

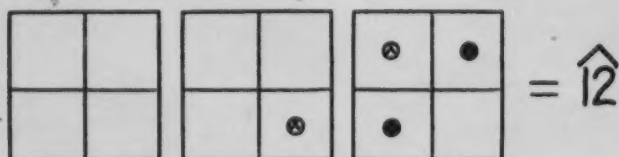
Exercise 3:

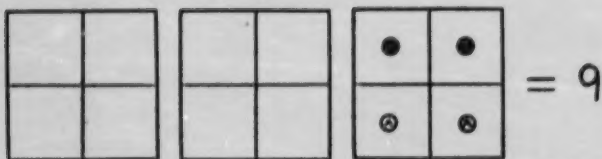


The red arrow could be for -9 .

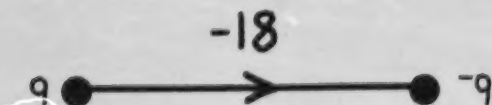


It's possible to transform $\hat{-3}$ to $\hat{-12}$ by moving the negative checker from the 1-square to the 10-square.



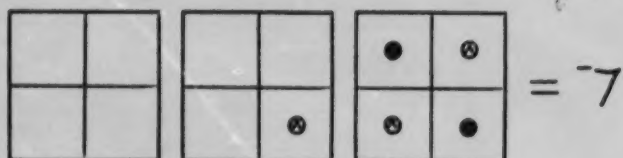
Exercise 4:

The red arrow could be for -18 .

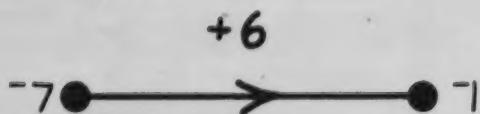


It's possible to transform 9 to -9 by moving the negative checker from the 2-square to the 20-square.



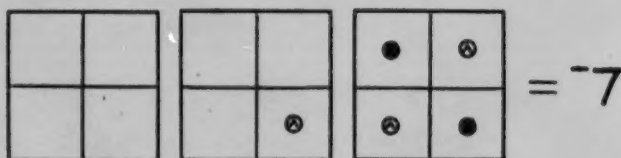
Exercise 5:

The red arrow could be for +6.



It's possible to transform -7 to -1 by moving the negative checker from the 10-square to the 4-square.



Exercise 6:

T: I would like to transform -7 to 7 by moving checkers on the Mini-computer. How much bigger is 7 than -7 ?

S: 14 bigger.

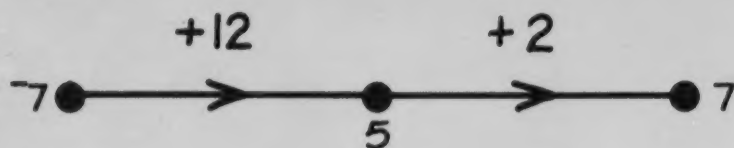
T: Is it possible to solve the problem by moving just one checker?

S: No.

T: By moving just two checkers?

S: Yes; we have to move a positive checker from the 8-square to the 20-square (+12 move) and a negative checker from the 4-square to the 2-square (+2 move).

As the students suggest these moves, progressively draw this arrow picture on the board:



Exercise 7:

Draw on the board:



Don't write the letter on the board. It is here just to simplify the description of this exercise.

	
$2 \times$	
$+8$	
$\frac{1}{2} \times$	
-4	

	
	$+9$
	-2
	$\times 2$
	$\div 2$

T: If the red arrow were for $2 \times$, what would this number be (point to a)?

S: -12 .

T: What could the blue arrow be for?

S: $+22$.

Write "+22" in blue in the chart.

T: If the blue arrow were for +9, what would this number be (point to a)?

S: 1

T: What could the red arrow be for?

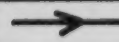
S: +7.

Write "+7" in red in the chart.

Have the students complete both charts by themselves. Encourage them to try to find many more solutions.

The following is one possible set of solutions:

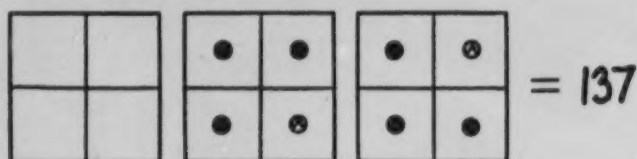
	
$2 \times$	$+22$
$+8$	$\times 5_{\text{or}} +8$
$\frac{1}{2} \times$	$+13$
-4	$+20$
$+7$	$\times 10_{\text{or}} +9$
$+15$	$+1$

	
$+7$	$+9$
$+18$	-2
$+11$	$\times 2$
$+26$	$\div 2$
$+11$	$+5$
$+6$	$+10$

Exercise 8:

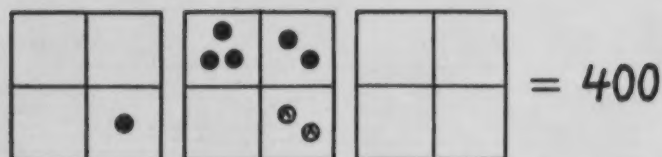
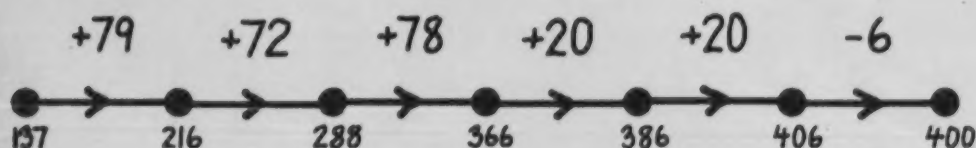
For the rest of the period, organize a game of Minicomputer Golf with positive and negative checkers. (For the procedure, see Exercise 4 in Section 3.2.)

You could start with



and choose 400 as a goal.

Here is the result that we got for this game in one of our fourth grade development classes.

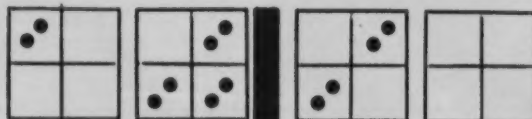


MULTIPLICATION

This chapter provides a rich variety of situations that reinforce and deepen the experiences of Part II of the CSMP Upper Primary Grades Program. Now, however, these experiences are broadened to include decimal numbers as multiplicands as well as fairly extensive preparation for the multiplication algorithm. For example,

Problems such as:

$$2 \times 87.60 =$$



$$2 \times 0.60 = 1.20$$

$$2 \times 7 = 14.00$$

$$2 \times 80 = 160.00$$

$$2 \times 87.60 = 175.20$$

lead to those like
$$\begin{array}{r} 245.40 \\ \times \quad 4 \\ \hline \end{array}$$
 where, instead of actually using

the Minicomputer, the students are encouraged to think as if they were. There is some exposure to the standard "carrying" notation. It should be strongly emphasized, however, that the purposes of this chapter do not include any emphasis on the paper and pencil algorithm, only exposure to systematic methods of multiplying.

CAPSULE LESSON SUMMARIES

Section 4.1: Review earlier experiences involving multiplication including those with decimal numbers. Using distributivity and the language of the Minicomputer, develop a systematic approach which leads to written forms such as:

$$\begin{array}{r} 789 \\ \times 4 \\ \hline 36 \\ 320 \\ 2800 \\ \hline 3,156 \end{array}$$

(Worksheet N06)

Section 4.2: Extend the situations of Section 4.1 to the decimal numbers. Introduce compact notation (standard "carrying" notation).

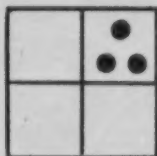
(Worksheets N07, N08, N09)

Section 4.3: Given a product (computed on a hand-calculator) such as $14 \times 15 = 210$, deduce related products such as 15×15 and 13×15 . Present calculations containing errors and encourage discussion of reasons for their incorrectness.

(Worksheets N10, N11, N12)

4.1 Multiplication Problems #1Exercise 1:

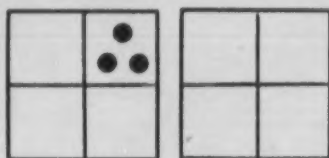
Put on the board:

T: What is this number? (Answer: $3 \times 4 = 12$)

Write on the board:

$$3 \times 4 = 12$$

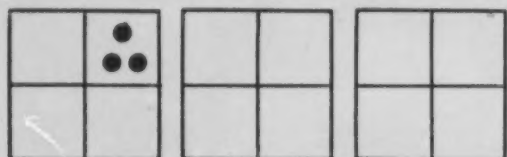
Put a second Minicomputer board to the right of the first one and ask the students to write (on a sheet of paper) the calculation that is now shown on the Minicomputer. (Answer: $3 \times 40 = 120$)



Write the answer on the board (below the previous calculation):

$$\begin{array}{rcl} 3 \times 4 & = & 12 \\ 3 \times 40 & = & 120 \end{array}$$

Put a third Minicomputer board to the right of the second one and ask the students to write (on a sheet of paper) the new calculation that is shown on the Minicomputer. (Answer: $3 \times 400 = 1,200$)



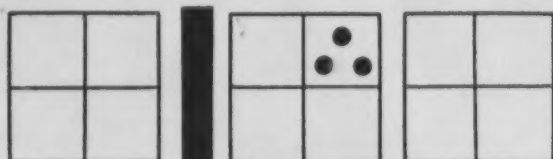
Write the answer on the board:

$$3 \times 4 = 12$$

$$3 \times 40 = 120$$

$$3 \times 400 = 1,200$$

Move the board from the far right to the far left and draw a green line in this way:



T: If one checker on this white square (point to the 1-square of the ones' board; i.e., the board to the left of the green line) is for \$1, how much money is now represented on the Minicomputer?

Have students whisper the answer in your ear (\$1.20).

Write on the board:

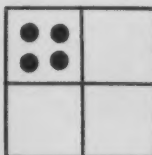
$$3 \times 0.40 = 1.20$$

or

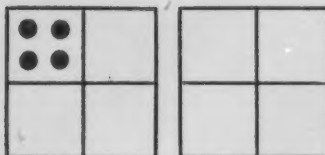
$$3 \times 0.4 = 1.2$$

Have students actually make the necessary robot moves^{*} to show that this answer is correct.

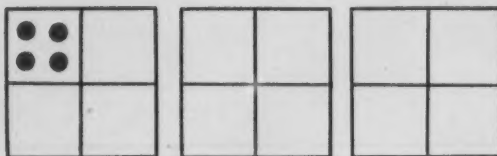
Follow similar steps for this sequence of activities:



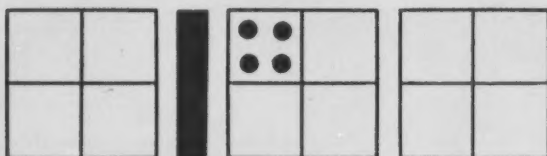
$$4 \times 8 = 32$$



$$4 \times 80 = 320$$



$$4 \times 800 = 3,200$$

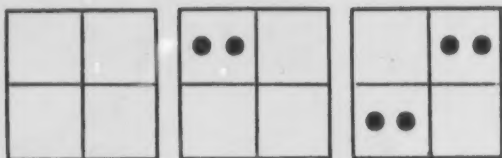


$$\begin{aligned} 4 \times 0.80 &= 3.20 \\ 4 \times 0.8 &= 3.2 \end{aligned}$$

^{*} Explain to your students that "robot moves" is the expression you will use from here on when referring to the standard forward and backward moves on the Minicomputer. This terminology is discussed more fully in the section "Standard Minicomputer Moves" of the introduction to the World of Numbers strand.

Exercise 2:

Put on the board:



T: The double of which number is on the Minicomputer? (Answer:
The double of 86)

Write on the board:

$$2 \times 86$$

Point to the ones' board and ask:

T: $2 \times 6 = ?$

S: 12

Point to the tens' board and ask:

T: $2 \times 80 = ?$

S: 160

Write on the board.

$$\begin{array}{rcl}
 2 \times 6 & = & 12 \\
 2 \times 80 & = & 160 \\
 \hline
 2 \times 86 & = &
 \end{array}$$

Have a student complete the calculation. (Answer: 172)

Follow similar steps for the following problems:

$$= 3 \times 484$$

$$3 \times 4 = 12$$

$$3 \times 80 = 240$$

$$3 \times 400 = 1,200$$

$$3 \times 484 = 1,452$$

$$= 4 \times 894$$

$$4 \times 4 = 16$$

$$4 \times 90 = 360$$

$$4 \times 800 = 3,200$$

$$4 \times 894 = 3,576$$

Exercise 3:

Write on the board:

$$\begin{array}{r} 376 \\ \times 2 \\ \hline \end{array}$$

T: We will solve this problem without actually using the Minicomputer any more. We will merely think of the Minicomputer. What number do we double on the ones' board?

S: We double 6. $2 \times 6 = 12$.

T: And on the tens' board?

S: We double 70. $2 \times 70 = 140$.

T: And on the hundreds' board?

S: We double 300. $2 \times 300 = 600$.

Write on the board:

$$\begin{array}{r} 376 \\ \times 2 \\ \hline 12 \\ 140 \\ 600 \\ \hline \end{array}$$

Have a student complete the calculation. (Answer: 752)

Follow similar steps for these problems:

$$\begin{array}{r}
 476 \\
 \times 3 \\
 \hline
 18 \\
 210 \\
 1,200 \\
 \hline
 1,428
 \end{array}$$

$$\begin{array}{r}
 568 \\
 \times 4 \\
 \hline
 32 \\
 240 \\
 2,000 \\
 \hline
 2,272
 \end{array}$$

Have the students solve these problems by themselves:

$$\begin{array}{r}
 275 \\
 \times 2 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 487 \\
 \times 3 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 789 \\
 \times 4 \\
 \hline
 \end{array}$$

(Answers: 550; 1,461; 3,156)

You can add some more for your faster students if you wish.

Exercise 4:

Draw this chart on the board and have the students copy it on a sheet of paper:

Item A: \$6	Item B: \$10	Item C: \$25	Item D: \$15
			9

T: You have \$250 to spend and there are four different kinds of items you can buy. The cost of each item is given in the chart. For example, each item C costs \$25. In each row of this chart, write the number of items of each kind that you could buy for a total cost of \$250. Many solutions are possible. Try to find at least one.

After a while, have a student give his or her idea.

Let's suppose the student says:

S: I shall buy nine of item C ($9 \times 25 = 225$) and one of item B and one of item D.

Write the numerals "0"; "1"; "9"; "1" in the first row of the chart. Verify with the students that the total amount spent in this case is \$250.

Have the students work on their own again.

Worksheet N06 is available for your faster students.

Before the end of the lesson, have some students complete the chart on the board. Here is a set of solutions that we got in one of our fourth grade development classes.

Item A: \$6	Item B: \$10	Item C: \$25	Item D: \$15
0	1	9	1
0	0	10	0
0	20	2	0
0	5	5	5
0	2	8	2
0	25	0	0
5	2	8	0
5	4	6	2
0	10	6	0
0	15	4	0
0	5	8	0
10	5	2	6
5	6	4	4
20	3	4	0
15	9	1	3
10	0	7	1

Name _____

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N06 **

In each row of this chart, write the number of items of each type that you could buy and spend \$1,000 altogether.

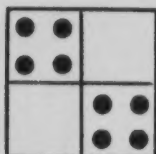
Item A: \$8	Item B: \$15	Item C: \$24	Item D: \$105
125	0	0	0
20	0	0	8
5	0	5	8
110	8	0	0
95	16	0	0
80	24	0	0
65	32	0	0
50	40	0	0
35	48	0	0
35	24	15	0

(Many other solutions are possible.)

4.2 Multiplication Problems #2

Exercise 1:

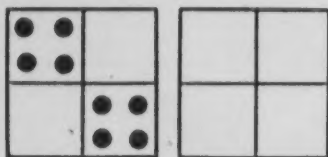
Put on the board:

T: What is this number? (Answer: $4 \times 9 = 36$)

Write on the board:

$$4 \times 9 = 36$$

Put a second Minicomputer board to the right of the first one and ask the students to write (on a sheet of paper) the calculation that is now shown on the Minicomputer. (Answer: $4 \times 90 = 360$)

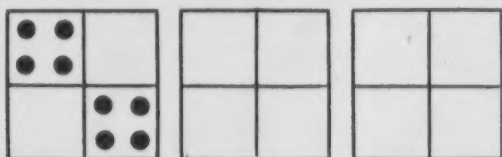


Write the answer on the board:

$$4 \times 9 = 36$$

$$4 \times 90 = 360$$

Put a third Minicomputer board to the right of the second one and ask the students to write (on a sheet of paper) the calculation that is now shown on the Minicomputer. (Answer: $4 \times 900 = 3,600$)



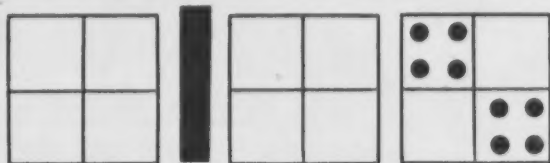
Write the answer on the board:

$$4 \times 9 = 36$$

$$4 \times 90 = 360$$

$$4 \times 900 = 3,600$$

Move the empty boards to the left and draw a green line in this way:



T: If one checker on this white square (point to the 1-square of the ones' board) is for \$1, how much money is now represented on the Minicomputer?

Have students whisper the answer in your ear (Answer: \$0.36) and write the complete calculation on their sheet of paper.

Write on the board:

$$4 \times 0.09 = 0.36$$

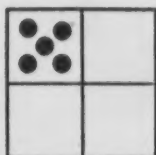
Ask a similar question for



$$4 \times 0.90 = 3.60$$

$$4 \times 0.9 = 3.6$$

Follow similar steps for this sequence of activities:



$$5 \times 8 = 40$$



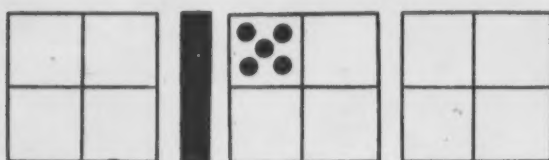
$$5 \times 80 = 400$$



$$5 \times 800 = 4,000$$



$$5 \times 0.08 = 0.40$$



$$5 \times 0.80 = 4.00$$

$$5 \times 0.8 = 4.0 = 4$$

10M

Write the following problems on the board and ask the students to solve them without using the Minicomputer.

$$2 \times 7 = 5 \times 6 =$$

$$2 \times 70 = 5 \times 60 =$$

$$2 \times 700 = 5 \times 600 =$$

$$2 \times 7,000 = 5 \times 6,000 =$$

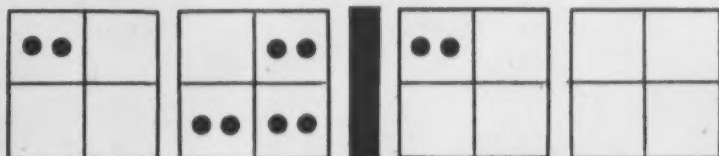
$$2 \times 0.70 = 5 \times 0.06 =$$

$$2 \times 0.07 = 5 \times 0.60 =$$

After a while, have several students write their answers on the board.

Exercise 2:

Put on the board:



T: The double of which number is on the Minicomputer? (Answer: the double of 87.80)

Write on the board:

$$2 \times 87.80$$

Point to the "dimes" board and ask:

T: $2 \times 0.80 = ?$

S: 1.60

Point to the ones' board and ask:

T: $2 \times 7 = ?$

S: 14

Point to the tens' board and ask:

T: $2 \times 80 = ?$

S: 160

Write on the board:

$$\begin{array}{rcl}
 2 \times 0.80 & = & 1.60 \\
 2 \times 7 & = & 14.00 \\
 2 \times 80 & = & 160.00 \\
 \hline
 2 \times 87.80 & = &
 \end{array}$$

Have a student complete the calculation. (Answer: 175.60)

Follow similar steps for the following problem:

$$= 4 \times 68.90$$

$$\begin{array}{rcl}
 4 \times 0.90 & = & 3.60 \\
 4 \times 8.00 & = & 32.00 \\
 4 \times 60.00 & = & 240.00 \\
 \hline
 4 \times 68.90 & = & 275.60
 \end{array}$$

Exercise 3:

Write on the board:

$$\begin{array}{r}
 576.60 \\
 \times 2 \\
 \hline
 \end{array}$$

T: We will solve this problem without actually using the Minicomputer any more. We shall merely think of the Minicomputer. What number do we double on the "dimes' " board?

S: We double 0.60. $2 \times 0.60 = 1.20$.

T: And on the ones' board?

S: We double 6. $2 \times 6 = 12$.

T: And on the tens' board?

S: We double 70. $2 \times 70 = 140$.

T: And on the hundreds' board?

S: We double 500. $2 \times 500 = 1,000$.

Write on the board:

$$\begin{array}{r}
 576.60 \\
 \times 2 \\
 \hline
 1.20 \\
 12.00 \\
 140.00 \\
 \underline{1,000.00}
 \end{array}$$

Have a student complete the calculation. (Answer: 1,153.20)

Follow similar steps for this problem:

$$\begin{array}{r}
 328.50 \\
 \times 3 \\
 \hline
 1.50 \\
 24.00 \\
 60.00 \\
 900.00 \\
 \hline
 985.50
 \end{array}$$

For the next problem, explain the usual compact writing to your students:

$$\begin{array}{r} \overset{\overset{1}{}}{\overset{\overset{1}{}}{}}{286.90} \\ \times 2 \\ \hline 573.80 \end{array}$$

Have the students solve these problems by themselves. Let them choose either the compact or the non-compact writing.

$$\begin{array}{r} 98.60 \\ \times 2 \\ \hline \end{array}$$

$$\begin{array}{r} 77.50 \\ \times 3 \\ \hline \end{array}$$

$$\begin{array}{r} 245.40 \\ \times 4 \\ \hline \end{array}$$

You can add other problems for your faster students if you wish.

Worksheets N07, N08, N09 are available for the rest of the period.

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Name _____ **N07** *

In each row of this chart, write the number of items of each type that you could buy and spend \$100 altogether. One solution is given for you.

Item A: \$5	Item B: \$8	Item C: \$12	Item D: \$25
6	1	1	2
11	1	1	1
16	1	1	0
8	6	1	0
0	11	1	0
4	1	6	0

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Name _____ **N08** **

In each row of this chart, write the number of items of each type that you could buy and spend \$300 altogether.

Item A: \$6	Item B: \$20	Item C: \$15	Item D: \$25
50	0	0	0
40	6	0	0
30	12	0	0
20	18	0	0
10	24	0	0
0	30	0	0
0	5	0	10
0	0	20	0

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Name _____ **N09** ***

In each row of this chart, write the number of items of each type that you could buy and spend \$1,000 altogether.

Item A: \$18	Item B: \$24	Item C: \$15	Item D: \$110
0	0	52	2
15	0	34	2
15	15	10	2
30	0	16	2
6	18	16	2
0	15	28	2
0	30	4	2
25	0	0	5
1	18	0	5

N07

(Many other solutions are possible.)

N08

(Many other solutions are possible.)

N09

(Many other solutions are possible.)

4.3 Multiplication Problems #3Exercise 1:

Write on the board:

$$6 \times 8 =$$

Ask a student to complete the calculation and write:

$$6 \times 8 = 48$$

$$7 \times 8 =$$

$$5 \times 8 =$$

T: How much bigger is 7×8 than 6×8 ?

S: 8 bigger.

T: So, if you have forgotten the calculation " 7×8 ," you can easily do it now. Explain how.

S: $7 \times 8 = 48 + 8 = 56$

T: How much smaller is 5×8 than 6×8 ?

S: 8 smaller

T: Then $5 \times 8 = ?$

S: $48 - 8 = 40$

T: I am going to write a calculation on the board that I got from my hand-calculator. (Be sure to show that you are using a hand-calculator.)

$$14 \times 15 = 210$$

Can you find an easy way to calculate "15 x 15" without using a hand-calculator?

Let the students react to this question. Probably some of them will suggest that 15 x 15 is 15 bigger than 14 x 15. Therefore
 $15 \times 15 = 210 + 15 = 225$.

Follow similar steps for:

$$13 \times 15 =$$

The students should notice that 13 x 15 is 15 less than 14 x 15. Therefore
 $13 \times 15 = 210 - 15 = 195$.

Accept any other suggestion of the class but emphasize the fact that 15 x 15 is 15 more than 14 x 15 and that 13 x 15 is 15 less than 14 x 15.

Write on the board:

$$27 \times 55 = 1,485$$

Ask the students to use this fact to calculate successively:

$28 \times 55 =$

$26 \times 55 =$

$29 \times 55 =$

$25 \times 55 =$

$30 \times 55 =$

$24 \times 55 =$

Exercise 2:

T: I shall write some calculations on the board and you are to solve them progressively. The first ones are easy, and gradually they become harder.

Write on the board:

$$10 \times 35 = \quad \quad 2 \times 35 =$$

Ask students to whisper the answers in your ear. Complete the calculations and write a third one on the board.

$$10 \times 35 = 350 \quad 2 \times 35 = 70$$

$$12 \times 35 =$$

T: Try to use the first two results to find the answer.
How did you solve this problem?

S: By adding 350 and 70; $12 \times 35 = 350 + 70 = 420$.

Write on the board:

$$20 \times 35 =$$

T: Again, try to use some of the results we already have to find the answer.

Ask students to explain their solutions. Some will probably notice that you already know that $2 \times 35 = 70$. So, $20 \times 35 = 10 \times 70 = 700$. Some others will perhaps suggest that since we know that $10 \times 35 = 350$, it follows that $20 \times 35 = 2 \times 350 = 700$. Accept any answer that is correct.

Write on the board:

$$\begin{array}{rcl} 20 \times 35 & = & 700 \\ 19 \times 35 & = & \end{array}$$

T: Is 19×35 larger or smaller than 700?

S: Smaller.

T: How much smaller?

S: 35 smaller.

T: Calculate " $700 - 35$." Solve this problem mentally.

S: $700 - 35 = 665$

Write on the board:

$$22 \times 35 =$$

T: What results could you use to make it easy to solve this problem?

S: $20 \times 35 = 700$; $2 \times 35 = 70$

T: What calculation must we do now?

S: $700 + 70 = 770$

Complete the calculation on the board. Write:

$$44 \times 35 =$$

T: How can we solve this new problem?

S: The easiest way is to double 770.

Ask the students to calculate " 2×770 " mentally and complete the calculation on the board. Write:

$$44 \times 35 = 1,540$$

$$45 \times 35 =$$

T: Is 45×35 larger or smaller than 1,540?

S: Larger.

How much larger?

S: 35 larger. So $45 \times 35 = 1,540 + 35 = 1,575$.

Write on the board:

$$44 \times 35 = 1,540$$

$$45 \times 35 = 1,575$$

$$43 \times 35 =$$

T: Is 43×35 larger or smaller than 1,540?

S: Smaller.

T: How much smaller?

S: 35 smaller. So $43 \times 35 = 1,540 - 35 = 1,505$.

Follow similar steps with:

$$100 \times 35 = 3,500$$

$$101 \times 35 = 3,500 + 35 = 3,535$$

$$99 \times 35 = 3,500 - 35 = 3,465$$

Exercise 3:

T: I am going to write some calculations on the board, but each of them is wrong. You must look at each one and without making a big calculation, tell me why it's wrong. You don't need to tell the right answer, just why it's wrong.

Write on the board:

$$11 \times 66 = 661$$

Let the students express their ideas. Emphasize the fact that $10 \times 66 = 660$ and that 11×66 should be 66 more than 660 and not just 1 more than 660.

Write on the board:

$$101 \times 56 = 5,596$$

Again, let the students discuss this incorrect calculation. Emphasize the fact that $100 \times 56 = 5,600$; that 101×56 is larger than 5,600 and therefore cannot be equal to 5,596 which is smaller than 5,600.

Write on the board:

$$99 \times 66 = 6,604$$

Let the students react. Some of them will probably notice that 99×66 is smaller than $100 \times 66 = 6,600$. So, 99×66 cannot be equal to 6,604, which is larger than 6,600.

Exercise 4:

Write on the board:

$$10 \times 38 = \quad \quad \quad 2 \times 38 =$$

Ask the students to complete these calculations.

Write on the board:

$$10 \times 38 = 380 \quad \quad 2 \times 38 = 76$$

$$\square \times 38 = 380 + 76$$

T: What number should be in the frame?

Probably most of the students will answer "456," which is, of course, incorrect.

Write:

$$\square \times 38 = 380 + 76 = 456$$

Repeat your question:

T: What number should be in the frame?

Be patient enough to wait until at least three or four students find the correct answer (12), then let one explain it to the whole group.

Follow similar steps for the following exercises:

- | | |
|--|--------------|
| ☐ $\times 38 = 380 + 38$ | (Answer: 11) |
| ☐ $\times 38 = 380 - 76$ | (Answer: 8) |
| ☐ $\times 38 = 380 + 76 + 76$ | (Answer: 14) |
| ☐ $\times 38 = 380 - 38$ | (Answer: 9) |

If your students are still interested, here are some other problems they may try:

- | | |
|---|---------------|
| ☐ $\times 38 = 3,800$ | (Answer: 100) |
| ☐ $\times 38 = 3,800 + 38$ | (Answer: 101) |
| ☐ $\times 38 = 3,800 + 380$ | (Answer: 110) |
| ☐ $\times 38 = 3,800 + 380 + 38$ | (Answer: 111) |

Worksheets N10, N11, N12 are available for the rest of the lesson.

Name _____

N10 *

Complete:

$10 \times 45 = \underline{450}$

$2 \times 45 = \underline{90}$

$12 \times 45 = \underline{540}$

$20 \times 45 = \underline{900}$

$22 \times 45 = \underline{990}$

$18 \times 45 = \underline{810}$

$40 \times 45 = \underline{1800}$

$39 \times 45 = \underline{1755}$

$80 \times 45 = \underline{3600}$

$100 \times 45 = \underline{4500}$

Name _____

N11 **

Complete:

$10 \times 54 = \underline{540}$

$3 \times 54 = \underline{162}$

$13 \times 54 = \underline{702}$

$30 \times 54 = \underline{1620}$

$43 \times 54 = \underline{2322}$

$60 \times 54 = \underline{3240}$

$59 \times 54 = \underline{3186}$

$100 \times 54 = \underline{5400}$

$103 \times 54 = \underline{5562}$

$130 \times 54 = \underline{7020}$

$133 \times 54 = \underline{7182}$

Name _____

N12 ***

$3 \times 46 = 138 \quad 10 \times 46 = 460$

Complete:

1) $460 \div 138 = \boxed{13} \text{ r } 46$

2) $460 \div 138 = \boxed{7} \text{ r } 46$

3) $460 \div 138 = 138 \div \boxed{16} \text{ r } 46$

4) $460 \div 460 = 460 \div \boxed{30} \text{ r } 46$

5) $3 \times 460 = \boxed{30} \text{ r } 46$

6) $460 \div 46 = \boxed{11} \text{ r } 46$

7) $460 \div 46 = \boxed{9} \text{ r } 46$

8) $4,600 \div \boxed{106} \text{ r } 46$

9) $4,600 \div 138 = \boxed{103} \text{ r } 46$

10) $4,600 \div 460 = \boxed{110} \text{ r } 46$

11) $4,600 \div 460 \div 138 = \boxed{113} \text{ r } 46$

INTRODUCTION TO THE RATIONAL NUMBERS^{*}

In a set of experiences involving measurement (in centimeters), division, multiples, and the notion of 1-steps, $\frac{1}{2}$ -steps, $\frac{1}{3}$ -steps, etc., the students have their first activity in marking points on a number line corresponding to rational numbers. In the process, the order among these numbers is suggested and many names for each of these is found.

CAPSULE LESSON SUMMARIES

Section 5.1: Find positions on the number line corresponding to various rational numbers. Explore several situations which emphasize the fact that a rational number may have many fractional names.

Section 5.2: Activities similar to those in Section 5.1 lead to the usual order relation on the rational numbers.

Section 5.3: Review Section 5.2 and present similar problems.

^{*}The rational numbers include the whole numbers and all numbers which have fractional names with integer numerators and positive integer denominators; e.g., $\frac{1}{2}$, $\frac{2}{3}$, $\frac{15}{75}$, $\frac{82}{103}$, $\frac{75}{12}$, $\frac{9}{5}$, $-\frac{3}{4}$, $-\frac{12}{20}$, $-\frac{8}{5}$, and so on. The decimal numbers are also rational numbers since, for example, $0.5 = \frac{1}{2}$; $1.3 = 1\frac{3}{10} = \frac{13}{10}$, and so on.

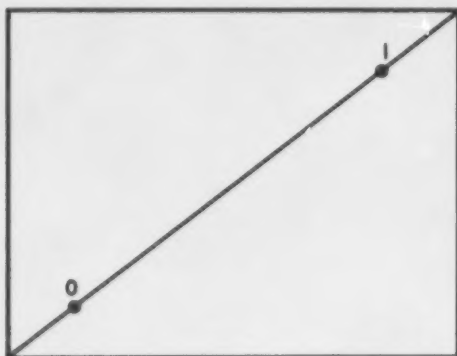
For communicating with the students we suggest following the usual custom of ignoring the distinction between rational numbers and their fractional names and thus to refer to them simply as "fractions." We think this is a better pedagogical practice at this level than using "rational numbers" even though this means that statements such as "0.5 is a fraction" must also be accepted as true for the time being.

5.1 Fractions #1

Distribute pieces of paper and 30 - centimeter rulers.

- T: Place your piece of paper lengthways in front of you and use your ruler to draw a line from one corner to another, going from the lower left to the upper right.

Next measure up the line 6 cm from the bottom corner and mark a dot on the line. Then measure up from that dot another 24 cm and mark a second dot. When you've done that you will have drawn part of the number line, because this is the dot for 0 and this is the dot for 1.



On this number line, if I start at 0 and go this way (toward 1) taking steps of length 24 cm, I will always land on the whole numbers and nothing else. If I go in the other direction from 0 taking steps of length 24 cm, I will land only on negative integers. Suppose I wanted to land on some fractions, what would I have to do?

- S: Take shorter steps.

- T: Suppose I wanted to land on all the halves: $\frac{1}{2}$, $\frac{3}{2}$, $\frac{6}{2}$, $\frac{12}{2}$, and so on, what size steps would I have to take?

S: Steps of length 12 cm.

T: How do you know?

S: $\frac{1}{2} \times 24 \text{ cm} = 12 \text{ cm}.$

T: Measure 12 cm up from the dot for 0 and mark a dot. Then measure up from that new dot 12 cm more. Where should you be?

S: At the dot for 1.

T: That's right. To get to the dot for 1, you need two half-steps. That's because $1 = \frac{2}{2}.$

T: Could I still land on all the halves if I took different size steps? Or can I do it only with 12 cm steps?

S: You would be able to land on the halves if you took 6 cm steps, but you'd land on some other dots besides those for the halves.

T: That's OK, so long as I land on all the halves. How do you know 6 cm steps would work?

S: Taking two of those steps is the same as taking a half-step.

S: You could take 4 cm steps too. Taking three of them is the same as taking a half-step.

T: How many half-steps does it take to get from 0 to $\frac{7}{2}$?

S: Seven.

T: How many 4 cm steps is that?

S: 21. There are three 4 cm steps to a half-step, and $3 \times 7 = 21.$

- T: Let's take a closer look at these different sorts of steps. We'll call a 24 cm step a "1 - step" because if you go up the number line in 24 cm steps the first place you land is the dot for 1. We've been calling a 12 cm step a " $\frac{1}{2}$ - step" because the first place you would land would be the dot for $\frac{1}{2}$.
What shall we call a 6 cm step?
What's the first place you land going up the number line from 0 taking 6 cm steps?

If no one is able to answer this question, tackle it from another angle.

- T: How many 6 cm steps does it take to get from the dot for 0 to the dot for 1?

S: Four.

- T: So how far along the way to the dot for 1 are you after the first 6 cm step?

S: One - fourth of the way. You're at the dot for $\frac{1}{4}$.

- T: So we call 6 cm steps, " $\frac{1}{4}$ - steps."

Proceed similarly to decide that 4 cm steps should be called " $\frac{1}{6}$ - steps."

- T: So we could land on all the halves if we took $\frac{1}{2}$ - steps, $\frac{1}{4}$ - steps, $\frac{1}{6}$ - steps.
Are there any other possibilities?

The children will start to pick up the pattern.

S: $\frac{1}{8}$ - steps . . . $\frac{1}{10}$ - steps . . . $\frac{1}{12}$ - steps . . .

- T: Take a look at all these numbers on the bottom of those fractions.
2, 4, 6, 8, 10, 12, . . . What kind of numbers are they?

S: Even numbers.

T: What's another name for them?

S: Multiples of 2.

T: So we could land on all the halves if we take "one-over-any-positive-multiple-of-2" steps.

Ask the children to turn over their sheets of paper, draw another diagonal, measure up 6 cm from the bottom corner and mark a dot, but this time measure up another 25 cm from this dot before marking a second dot. Once again these are the dots for 0 and 1, respectively.

T: On this number line, how long is a 1-step?

S: 25 cm.

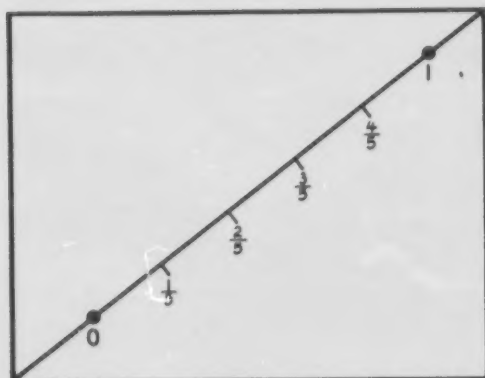
T: Suppose I want to start at 0 and land on all the fifths. What kind of steps should I take?

S: $\frac{1}{5}$ - steps.

T: How long would they be?

S: 5 cm, because $\frac{1}{5} \times 25 \text{ cm} = 5 \text{ cm}$.

T: On your copy of the number line, mark all the places you would land going up the number line from 0 taking $\frac{1}{5}$ - steps. Write the names of the numbers next to their dots.



If anyone has difficulty naming the dots you could perhaps provide assistance as follows:

$\frac{1}{5}$ is the first place you land ;

$\frac{2}{5}$ is the second place you land ;

and so on.

T: Is there any other kind of step you could take so as to land on all the fifths?

S: You could take two steps for every $\frac{1}{5}$ - step.

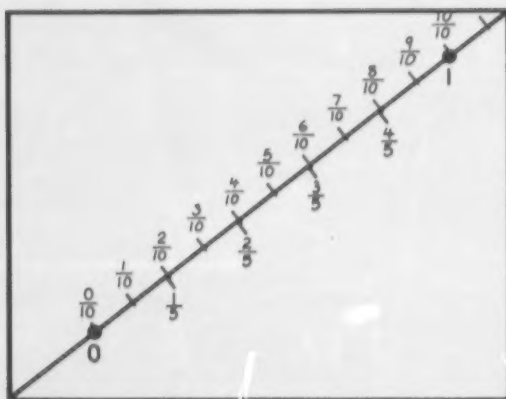
T: What kind of step would that be?

S: $\frac{1}{10}$ - step.

T: How long would they be?

S: $2\frac{1}{2}$ cm or 2.5 cm.

T: Use a different color and show all the places on your number line where you would land going up the number line from 0 in $\frac{1}{10}$ - steps.



T: How many $\frac{1}{5}$ - steps does it take to get to $\frac{9}{5}$ from 0?

S: Nine.

T: How many $\frac{1}{10}$ - steps does it take to get to $\frac{9}{5}$ from 0?

S: 18, because there are two $\frac{1}{10}$ - steps for every $\frac{1}{5}$ - step and $2 \times 9 = 18$.

Write " $\frac{9}{5} = \frac{18}{10}$ " on the board.

T: What other kind of steps could you take and still land on all the fifths?

S: $\frac{1}{15}$ - steps.

Continue in this way until you have

$$\frac{9}{5} = \frac{18}{10} = \frac{27}{15} = \frac{36}{20}$$

written on the board and the children have realized you can reach all the fifths by taking "one - over - any - positive - multiple - of - 5" steps.

9M

T: So we can land on the halves by taking "one-over-any-positive-multiple-of-2" steps; and we can land on the fifths by taking "one-over-any-positive-multiple-of-5" steps. What kind of steps do you think we should take to land on the thirds?

S: "One-over-any-positive-multiple-of-3" steps.

Ask similar questions about the fourths, the sixths and the eighths.

5.2 Fractions #2

Distribute pieces of paper and 30 - centimeter rulers.

Ask the children to draw a diagonal on their sheet of paper, mark a dot for 0 eight centimeters up from the lower left hand corner, and a dot for 1 twenty centimeters up from the dot for 0.

T: How long is a 1 - step on this number line?

S: 20 cm.

T: How long is a $\frac{1}{4}$ - step on this number line?

S: 5 cm, because $\frac{1}{4} \times 20 \text{ cm} = 5 \text{ cm}$.

T: How many $\frac{1}{4}$ - steps does it take to get from 0 to $\frac{3}{4}$?

S: Three.

T: How far is that?

S: 15 cm; $3 \times 5 \text{ cm} = 15 \text{ cm}$.

T: Mark a dot for $\frac{3}{4}$ on your number line.
How many $\frac{1}{4}$ - steps does it take to get from 0 to $-\frac{1}{4}$?

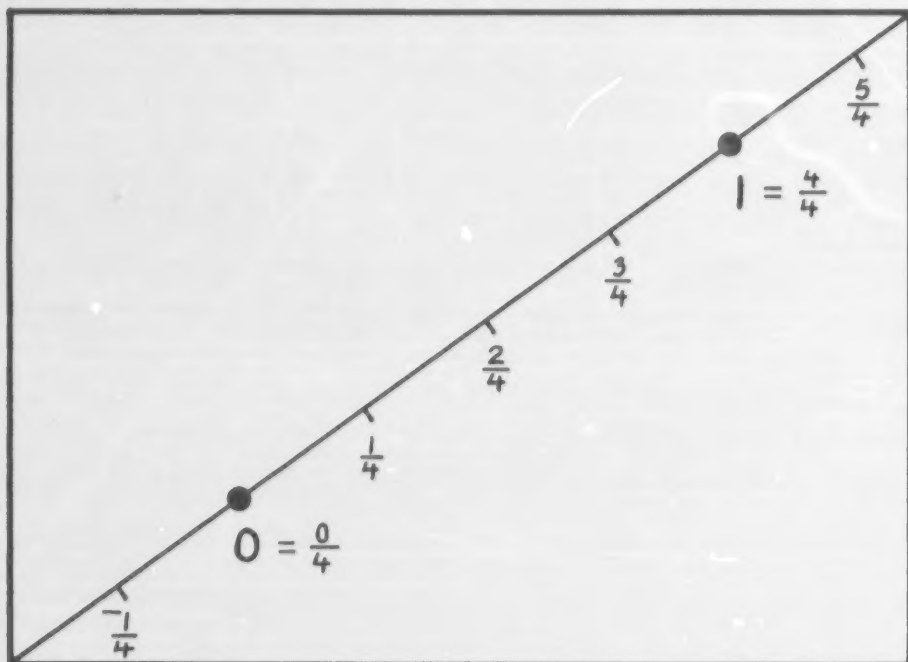
S: One, but you have to go down instead of up.

T: How far down from the dot for 0 is that?

S: 5 cm.

T: Mark a dot for $-\frac{1}{4}$ on your number line. Then mark dots for these other numbers:

$$\frac{5}{4}, \frac{2}{4}, \frac{4}{4}, \frac{0}{4}, \frac{1}{4}.$$



T: What other kinds of steps could we take and still land on all the fourths?

S: $\frac{1}{8}$ -steps ... $\frac{1}{12}$ -steps ... $\frac{1}{16}$ -steps ... "one-over-any-positive-multiple-of-4" steps.

T: Let's suppose you take $\frac{1}{12}$ -steps. How many will you need to take to get to $\frac{5}{4}$?

S: 15. There are three $\frac{1}{12}$ -steps to every $\frac{1}{4}$ -step. It takes five $\frac{1}{4}$ -steps to get to $\frac{5}{4}$ from 0; so it will take $3 \times 5 = 15$ $\frac{1}{12}$ -steps to get there.

Write " $\frac{5}{4} = \frac{15}{12}$ " on the board.

T: Now suppose I want to start at 0 and land on all the sixths.
What kind of steps should I take?

S: "One-over-any-positive-multiple-of-6" steps.

T: What is the longest kind of step you could take and still land on all the sixths?

S: $\frac{1}{6}$ - steps.

T: How many $\frac{1}{6}$ - steps does it take to get to $\frac{2}{6}$?

S: Two.

T: Which of $\frac{1}{4}$ and $\frac{2}{6}$ do you think is the bigger?
Or do you think they're equal?
How can we find out for sure?

S: Find some kind of step that will land on both the fourths and the sixths, and then see which one you get to first when you're going up the line.

T: That's a good idea. What kind of steps should we take to land on all the fourths?

S: One-over-any-positive-multiple-of-4.

Write on the board, with the children's help:

4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, . . .

T: What kind of steps should we take to land on all the sixths?

S: One - over - any - positive - multiple - of - 6.

Again, use the children's help to write on the board:

6, 12, 18, 24, 30, 36, 42, . . .

T: So what kind of steps do we need to land on both the fourths and the sixths?

S: $\frac{1}{12}$ - steps.

T: Any others?

S: 24, 36, . . .

T: What would the next one be?

S: 48.

T: What kind of numbers are they?

S: Multiples of 12.

T: OK. Back to our problem. How many $\frac{1}{12}$ - steps to get to $\frac{1}{4}$?

S: Three.

Write " $\frac{3}{12} = \frac{1}{4}$ " on the board.

T: How many $\frac{1}{12}$ - steps to get to $\frac{2}{6}$?

S: You need two $\frac{1}{6}$ - steps to get to $\frac{2}{6}$. Each $\frac{1}{6}$ - step is made up of two $\frac{1}{12}$ - steps. So that means you need $2 \times 2 = 4$ $\frac{1}{12}$ - steps.

Write " $\frac{4}{12} = \frac{2}{6}$ " on the board.

T: So which is the bigger: $\frac{1}{4}$ or $\frac{2}{6}$?

S: $\frac{2}{6}$ because it needs four $\frac{1}{12}$ -steps and $\frac{1}{4}$ only needs three.

T: Now I'm going to write some pairs of numbers on the board. In each pair you have to decide whether one is bigger than the other or if they're both the same. You are to indicate what you think the answer is by placing one of the signs "<", ">" and "=" between the two numbers. You may choose which level of difficulty you like.

NOTE: The answers to each of the problems are given in red. You should copy the chart on the board as it appears below with everything in red omitted.

★	★★	★★★
$\frac{6}{12} > \frac{1}{4}$	$\frac{11}{12} > \frac{5}{6}$	$\frac{7}{3} > \frac{27}{12}$
$\frac{3}{6} > \frac{1}{4}$	$\frac{4}{6} > \frac{7}{12}$	$\frac{18}{6} = \frac{12}{4}$
$\frac{5}{12} < \frac{4}{6}$	$\frac{5}{4} = \frac{15}{12}$	$\frac{49}{6} > 8$

Before ending the lesson, go over at least the one-star problem with the whole class.

5.3 Fractions #3

Review the method of comparing rational numbers introduced in Section 5.2 by comparing the following pairs of numbers:

$$\frac{3}{5} \quad \frac{2}{3} \quad (\text{Answer: } \frac{3}{5} < \frac{2}{3})$$

$$\frac{1}{3} \quad \frac{5}{12} \quad (\text{Answer: } \frac{1}{3} < \frac{5}{12})$$

$$\frac{1}{4} \quad \frac{2}{10} \quad (\text{Answer: } \frac{1}{4} > \frac{2}{10})$$

For the remainder of the lesson let the children work individually with comparisons chosen from the following three categories.

NOTE: The answers to each of the problems are given in red. You should copy the chart on the board as it appears below with everything in red omitted.

★	★★	★★★
$\frac{1}{3} < \frac{1}{2}$	$\frac{5}{6} < \frac{7}{8}$	$\frac{11}{13} < \frac{6}{7}$
$\frac{2}{4} = \frac{4}{8}$	$\frac{7}{12} > \frac{5}{9}$	$-\frac{4}{9} > -\frac{7}{15}$
$\frac{4}{6} > \frac{1}{2}$	$-\frac{3}{4} > -\frac{5}{6}$	$\frac{-1976}{1776} < \frac{2000}{1976}$
$\frac{3}{5} < \frac{7}{10}$	$\frac{11}{9} > \frac{7}{6}$	$\frac{17}{7} > \frac{19}{9}$

6

DIVISION

The lessons in this chapter reinforce and extend the experiences of Part II of the CSMP Upper Primary Grades Program. On the surface, there seems little that is new. But these lessons probe much more deeply towards explicit experiences that later lead to the division algorithm and encourage the use of arrow diagrams as tools for solving division problems.

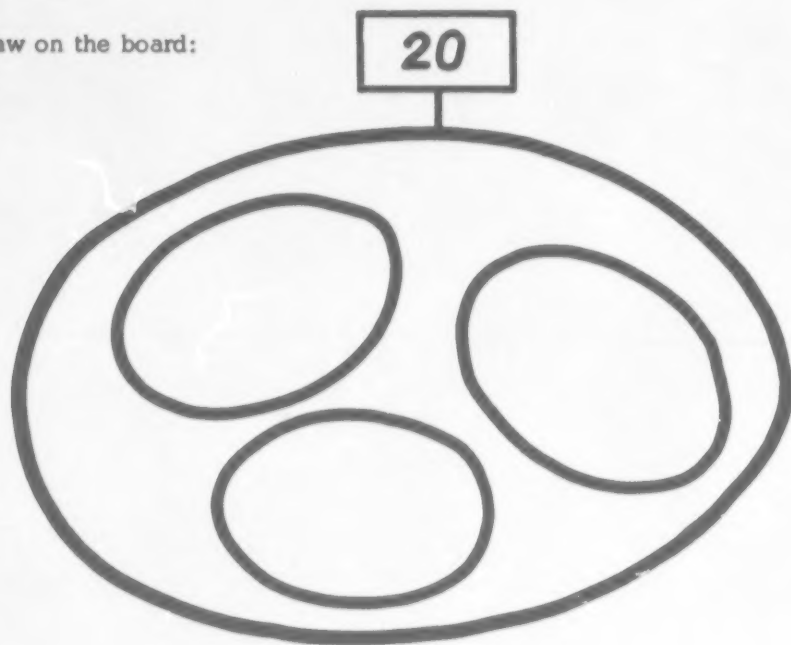
CAPSULE LESSON SUMMARIES

Section 6.1: Use arrow diagrams to solve division problems which present themselves in several real situations.
(Worksheets N13, N14, N15)

Section 6.2: Present a road building problem and use the result to solve a division problem. Explore several similar situations.

6.1 Division Problems #1Exercise 1:

Draw on the board:



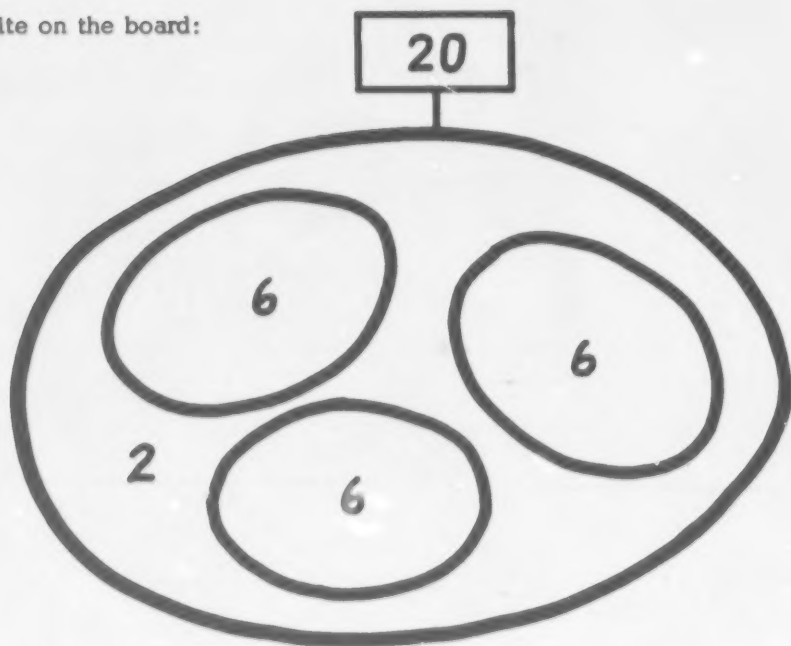
T: There are exactly 20 nuts inside the black string. You have to put the same number of nuts in each red string and you have to put as many nuts in the red strings as possible.

Have the students whisper the answer in your ear (six nuts in each red string; two nuts are left over).

T: There are exactly 20 nuts inside the black^{*} string ...

^{*} In this and similar situations, for the word "black" you should substitute "white" or whichever color of chalk you are actually using. You would not, of course, use dark-colored chalk on a dark-colored chalkboard.

Write on the board:



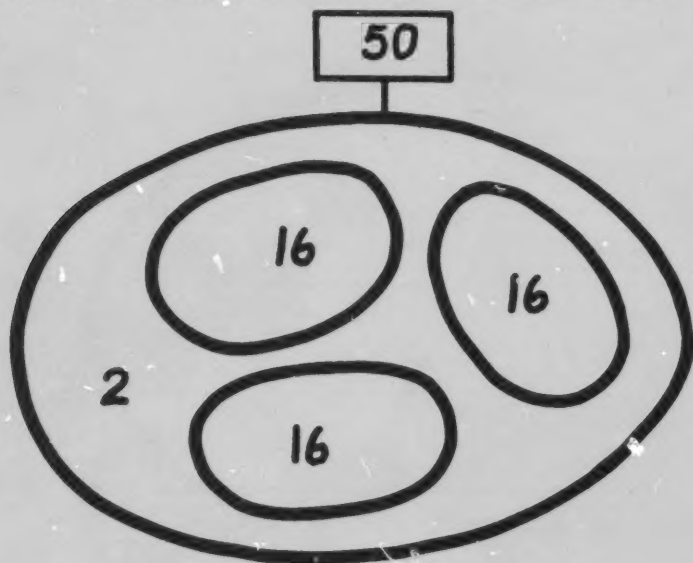
T: Write a number story for this picture.

Accept any answer that is correct. But emphasize this calculation:

$$(3 \times 6) + 2 = 20$$

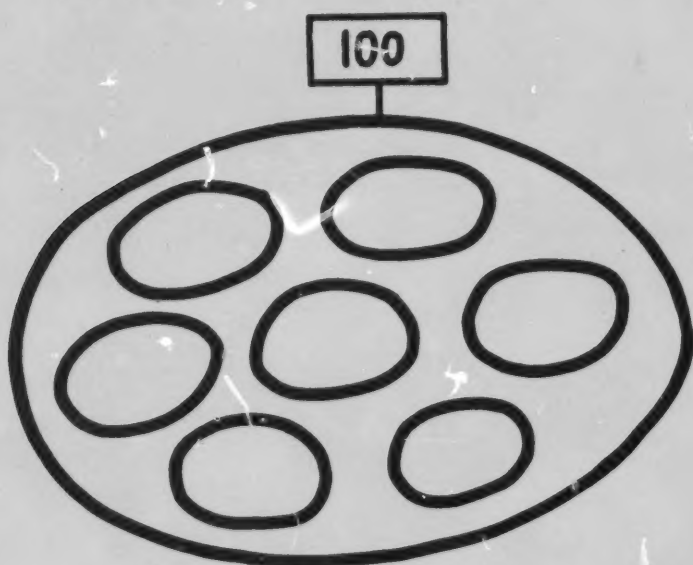
Erase the numbers from the picture on the board and write "50" in the tag. Ask the students to solve the new problem on a sheet of paper. (Same number of nuts inside each red string; as many nuts as possible in each of the red strings.)

After a while have a student write the answer on the board:



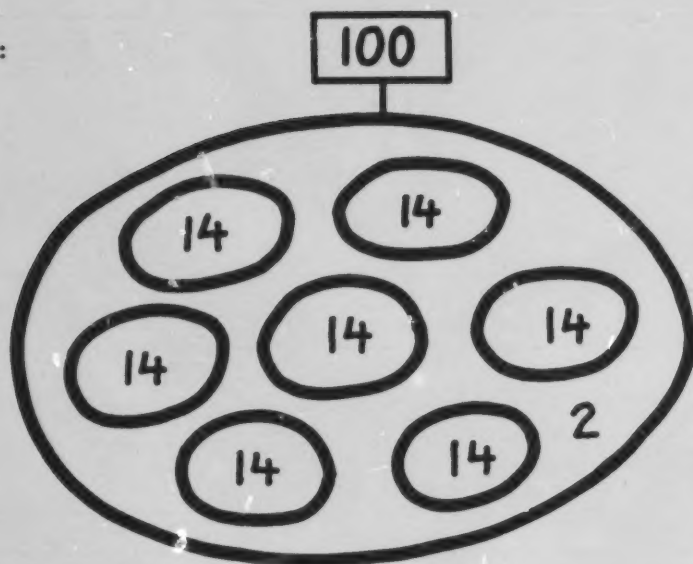
$$(3 \times 16) + 2 = 50$$

Follow similar steps for this problem:



168

Answer:



$$(7 \times 14) + 2 = 100$$

Pass out worksheets N13, N14, N15 and have the students work on their own. After a while, draw the pictures of worksheets N13 and N14 on the board and have the students complete them.

Exercise 2:

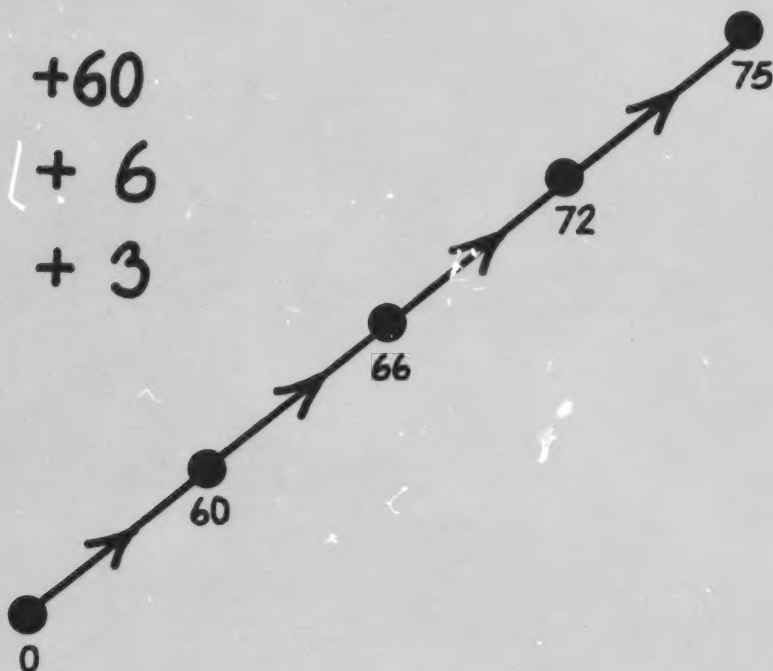
T: We have 75 bottles which we are putting in cartons. Each carton holds exactly 6 bottles. How many cartons can we fill completely and how many bottles will we have left over?
Use an arrow picture to solve this problem if you wish.

Write on the board:

75 bottles.
Each carton holds exactly 6 bottles.

Have the students solve the problem on a sheet of paper. Walk around and observe their answers.

Then draw on the board:



T: This arrow picture shows a way of solving the problem.
Can you explain it?

Have the students express their ideas about the picture.

T: What does the red arrow mean?

S: 60 bottles.

T: In how many cartons?

S: 10 cartons.

T: What do the blue arrows mean?

S: Two more cartons.

T: And the green arrow?

S: Three bottles are left over.

T: How many cartons did we fill completely?

S: 12 cartons.

Write on the board:

$$(12 \times 6) + 3 = 75$$

Write on the board:

260 bottles.	★
Each carton holds 6 bottles.	

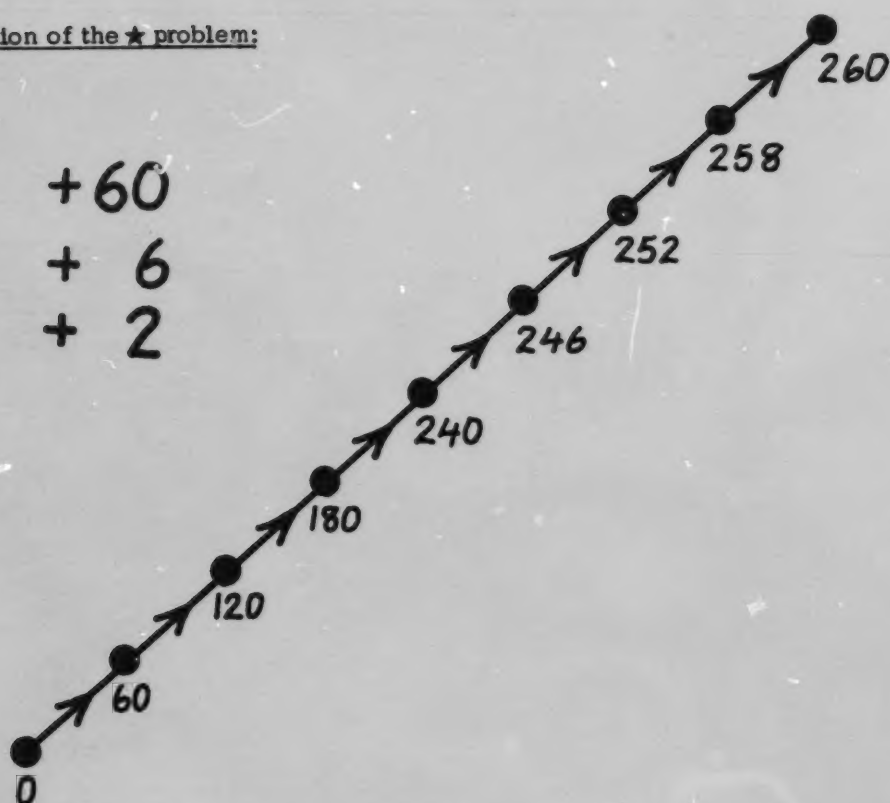
T: Solve this new problem by drawing a similar arrow picture.

For the students who solve this problem quickly, write this two-star problem on the board:

525 bottles.	★★
Each carton holds 12 bottles.	

Before the end of the lesson, have two students draw the solutions for the problems on the board and have other students explain them.

Solution of the ★ problem:



Comments:

Each red arrow shows that we have filled 10 cartons with 60 bottles.

The four red arrows show that we have filled 40 cartons with 240 bottles.

The three blue arrows show that we have filled three more cartons with 18 bottles.

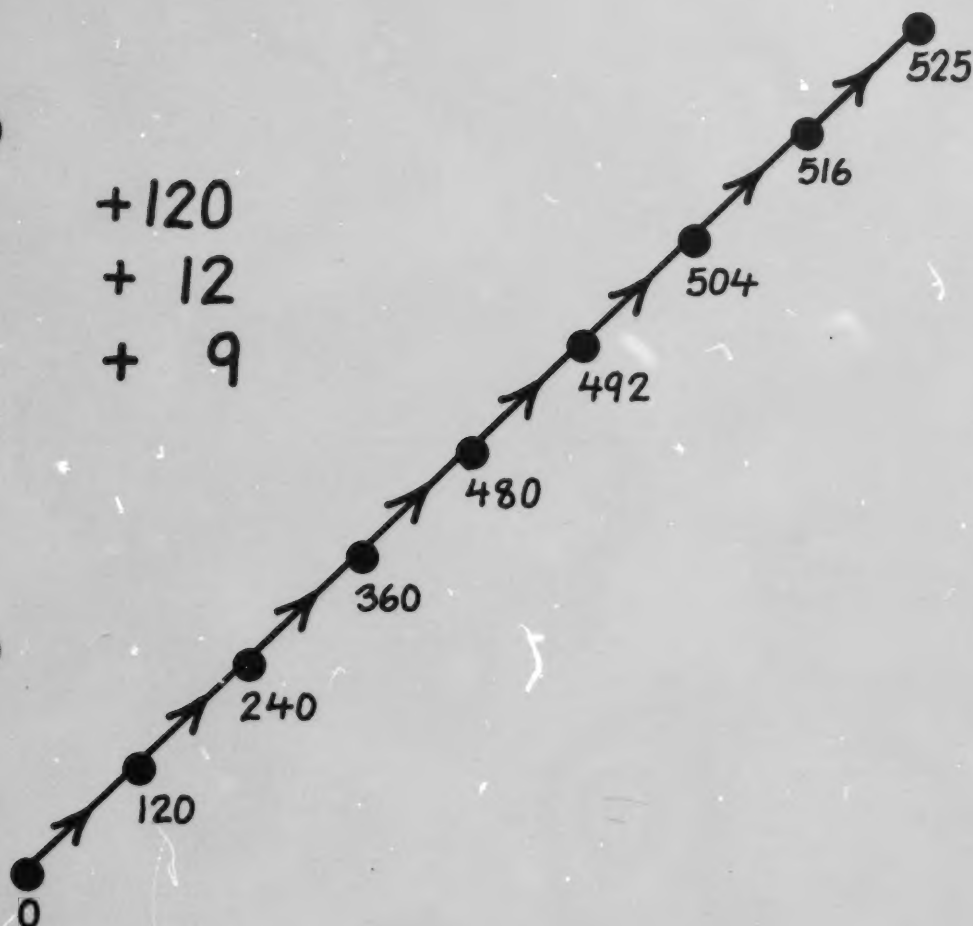
Altogether 43 cartons are filled with 258 bottles and two bottles are left over.

A number story for this picture could be:

$$(43 \times 6) + 2 = 260$$

Solution of the ★ ★ problem:

$$\begin{array}{r} + 120 \\ + 12 \\ + 9 \end{array}$$



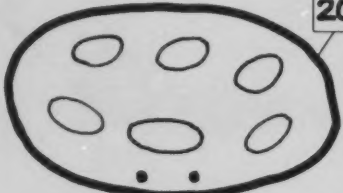
This arrow picture can be explained in a way similar to the preceding one.
One number story could be:

$$(43 \times 12) + 9 = 525$$

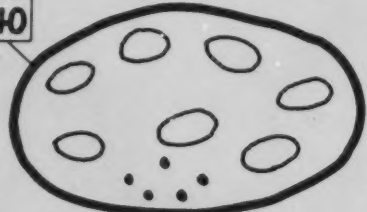
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Name _____

N13 *



Calculation: $(3 \times 6) + 2 = 20$

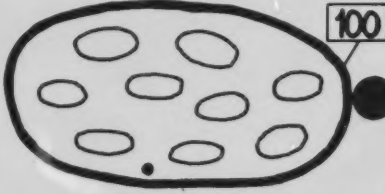


Calculation: $(5 \times 7) + 5 = 40$

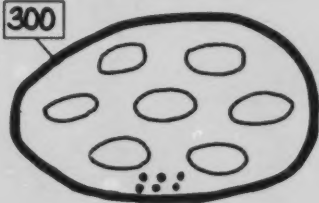
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Name _____

N14 **



Calculation: $(11 \times 9) + 1 = 100$

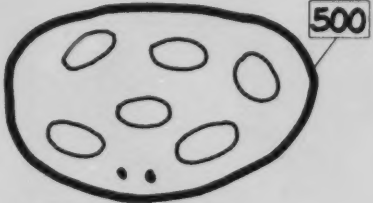


Calculation: $(42 \times 7) + 6 = 300$

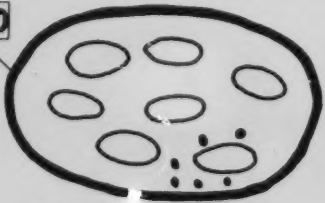
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Name _____

N15 ***



Calculation: $(83 \times 6) + 2 = 500$

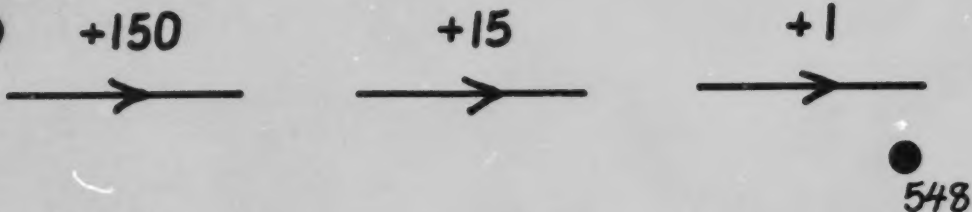


Calculation: $(142 \times 7) + 6 = 1,000$

6.2 Division Problems #2

Exercise 1:

Draw on the board:



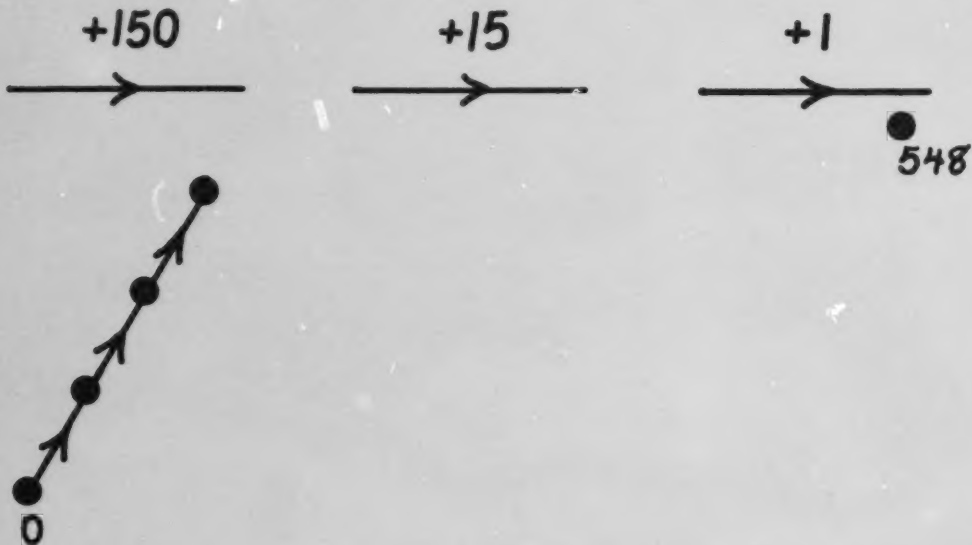
0

T: Build a red-blue-green road from 0 to 548. Try to use as few arrows as possible.

Have the students work on their own for a short period of time. Watch the children's pictures.

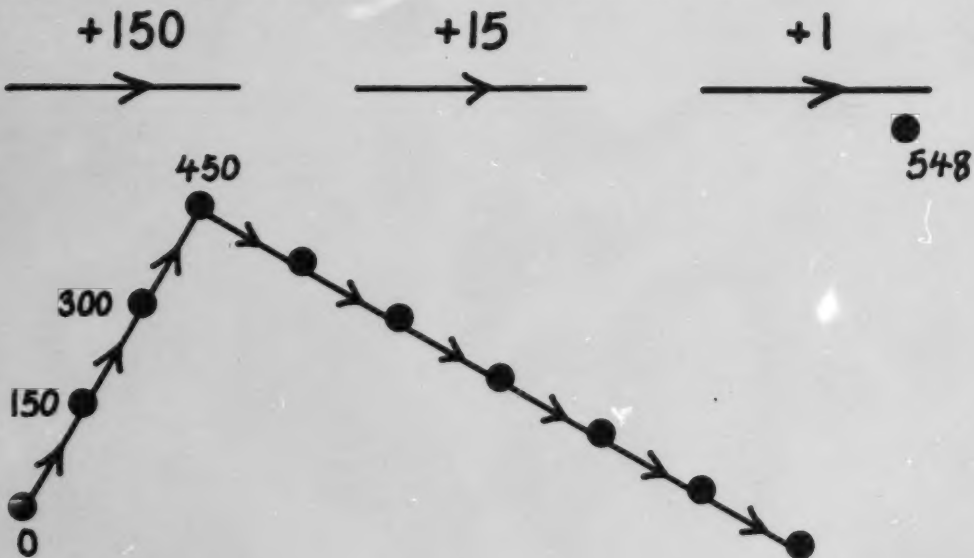
T: Most of you have drawn red arrows first!
How many of them did you draw? (Answer: 3)

Draw them on the board and ask students to label the dots.



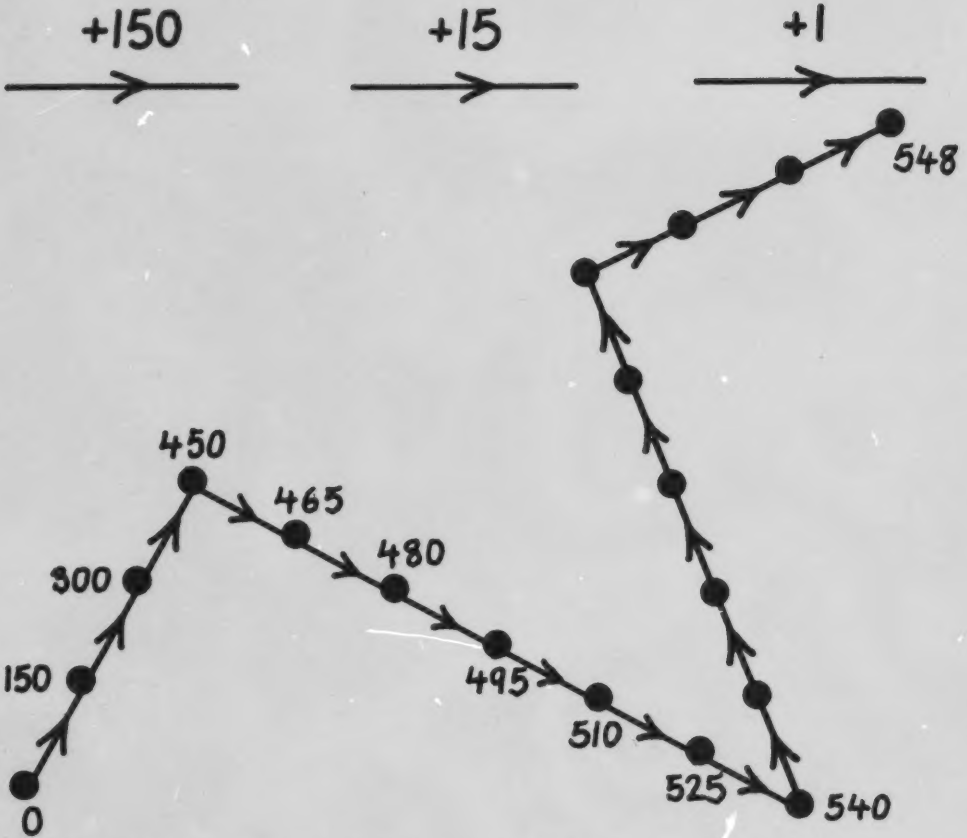
T: How many blue arrows did you draw? (Answer: 6)

Draw them on the board and ask students to label the dots.



T: How can we complete the picture? (Answer: by drawing eight green arrows)

Draw them on the board:



T: Let's pretend that you have to put 548 apples in boxes. Each box can hold 15 apples.

How many boxes can you fill completely?

How many apples will you have left over?

Write on the board:

548 apples Each box can hold 15 apples
--

T: You can solve this problem with the help of the arrow picture.
Write your answer on a sheet of paper.

When several students have solved the problem (36 boxes are filled up; eight apples are left over), point to the first red arrow and its ending dot.

T: How many boxes do you need to hold these 150 apples?

S: 10 boxes.

T: How do you know?

S: $10 \times 15 = 150$

T: Each time you draw a red arrow, it means that you have filled 10 boxes. After drawing three red arrows, how many boxes did you fill?

S: 30 boxes.

T: And how many apples did you put in the 30 boxes?

S: 450 apples.

Point to the six blue arrows:

T: From 450 to 540, how many more apples?

S: 90 apples.

T: How many boxes did you use?

S: Six boxes.

T: How many boxes did you fill altogether?

S: 36 boxes.

T: How many apples are left?

S: Eight apples.

T: Now I shall write some calculations on the board. You can find the answer to each of them on the arrow picture.

Write:

$$20 \times 15 = \underline{\hspace{2cm}}$$

Ask a student to show the dot for the answer (300) on the board.

Follow similar steps for:

$$30 \times 15 = \underline{\hspace{2cm}} \quad (\text{Answer: } 450)$$

$$32 \times 15 = \underline{\hspace{2cm}} \quad (\text{Answer: } 480)$$

$$35 \times 15 = \underline{\hspace{2cm}} \quad (\text{Answer: } 525)$$

$$36 \times 15 = \underline{\hspace{2cm}} \quad (\text{Answer: } 540)$$

$$(36 \times 15) + \underline{\hspace{2cm}} = 548 \quad (\text{Answer: } 8)$$

T: Is the number 37×15 on the board?

S: No.

T: By drawing one more arrow, can you show this number on the board?

S: Yes, by drawing a blue arrow starting from 540. Its endpoint is 37×15 .

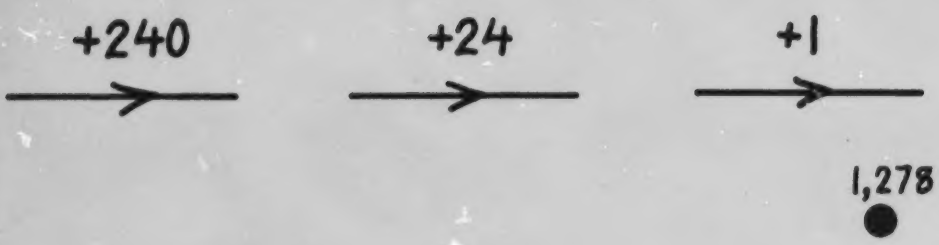
T: Is the number 40×15 on the board?

S: No.

- T: By drawing one more arrow, can you show this number on the board?
- S: Yes, by drawing a red arrow starting from 450. Its endpoint is 40×15 .

Exercise 2:

Draw on the board:

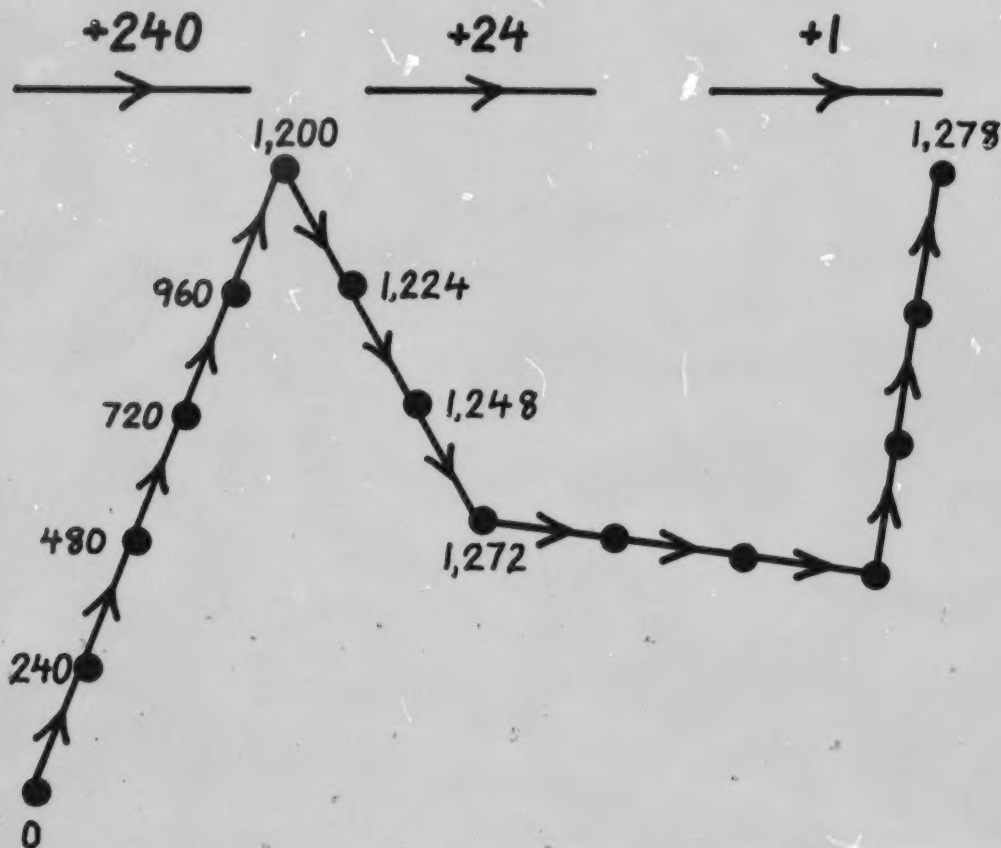


- 0
- T: Draw a red-blue-green road from 0 to 1,278. Try to use as few arrows as possible.

180

205

Let the students work on their own for a while. Then draw the road on the board with the help of the class.



T: You have to put 1,278 bottles in boxes. Each box can hold 24 bottles. How many boxes can you fill completely?
How many bottles will you have left over?

Write on the board:

1,278 bottles
Each box can hold 24 bottles

T: Try to solve this problem with the help of the arrow picture.
Write your answer on a sheet of paper.

When several students have found the answer (53 boxes are filled up; six bottles are left over), ask one of them to explain his or her solution to the whole group.

Write the following calculations on the board and for each of them, ask a student to show the dot for the answer on the board:

$$10 \times 24 = \underline{\quad\quad} \quad (\text{Answer: } 240)$$

$$30 \times 24 = \underline{\quad\quad} \quad (\text{Answer: } 720)$$

$$50 \times 24 = \underline{\quad\quad} \quad (\text{Answer: } 1,200)$$

$$52 \times 24 = \underline{\quad\quad} \quad (\text{Answer: } 1,248)$$

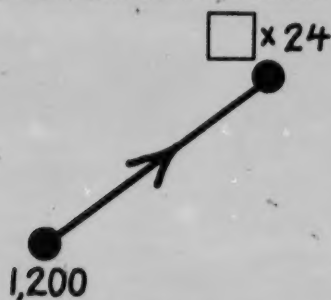
$$53 \times 24 = \underline{\quad\quad} \quad (\text{Answer: } 1,272)$$

Ask the students to complete the following calculations with the help of the arrow picture:

$$960 = \square \times 24 \quad (\text{Answer: } 40)$$

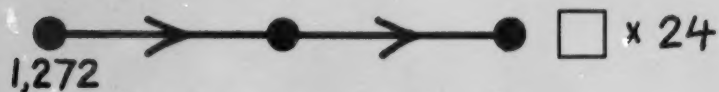
$$1,224 = \square \times 24 \quad (\text{Answer: } 51)$$

On the board draw one more red arrow starting from 1,200:



T: What number should be put in the frame? (Answer: 60)

On the board draw two more blue arrows starting from 1,272:



T: What number should be put in the frame? (Answer: 55)

F

WHAT COULD THESE ARROWS BE FOR ?

The language of arrows is used to present a variety of open-ended numerical situations. For the most part, these involve the use of various combinations of numerical functions which are used to solve problems given certain data. Two examples follow:

Example 1:

Given a choice of five numerical functions and five arrow diagrams, which functions could be used to label which arrow diagrams?

$$\boxed{\times 2} \quad \boxed{-3} \quad \boxed{-10} \quad \boxed{\times 5} \quad \boxed{\div 2}$$



Example 2:

Given:



the students are to complete the following chart:

$\times 4$	
-4	
	$+10$
	-10

Situations similar to those above engage the student in numerical activities at various levels. There is a great deal of numerical calculation involved. There are many opportunities for the growth of understanding of

the interrelationships among numerical operations and functions. There are many situations which require the student to develop short sequences of arguments: "If the red arrow is for _____, then the blue arrow could be for _____."

CAPSULE LESSON SUMMARIES

Section 1.1: Match collections of numerical functions with partially labeled single-colored arrow diagrams.

(Worksheets N16, N17, N18)

Section 1.2: Present various partially labeled arrow diagrams involving a pair of numerical functions. First, students match pairs of functions which could be used to label arrow diagrams. Then one of two numerical functions is given or the choice of a pair of functions is left completely open, and students find ways to complete the labeling.

(Worksheets N19, N20)

Section 1.3: Extend the activities of Section 1.2 to more complex situations.

(Worksheets N21, N22, N23)

Section 1.4: Explore such numerical relations as, "I am a multiple of you", "I am larger than you" and "I am at least 10 smaller than you".

(Worksheets N24, N25, N26, N27)

1.1 Tag the Arrow Pictures

If you have two chalkboards, draw the pictures from this page and page 5 on them before the lesson.

Exercise 1:

Draw on the board:

$\times 2$

-3

-10

$\times 5$

$\div 2$



Figure 1

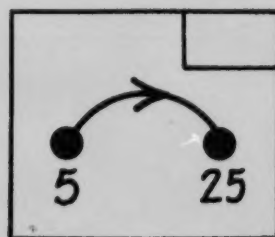


Figure 2

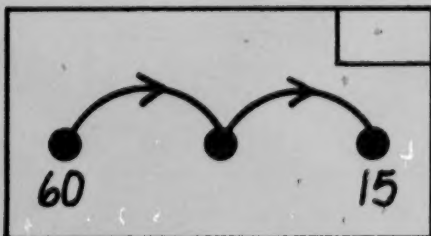


Figure 3

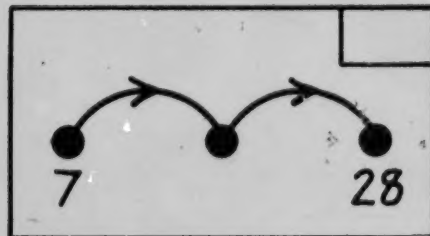


Figure 4

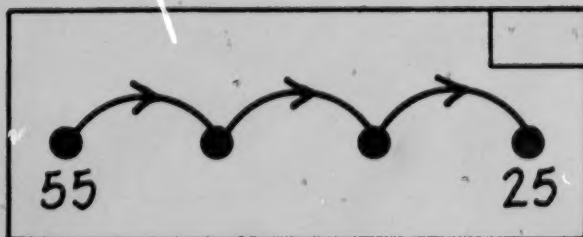


Figure 5

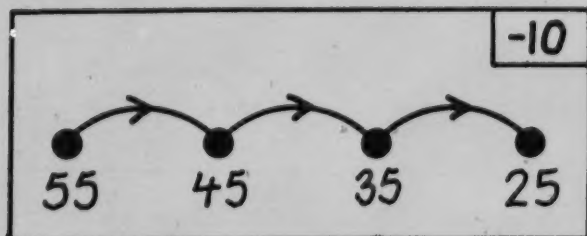
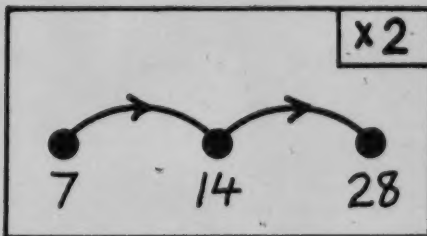
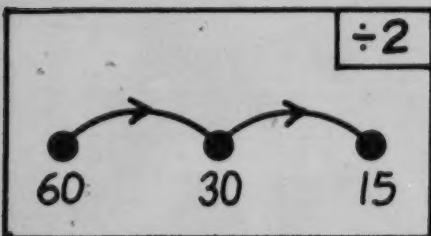
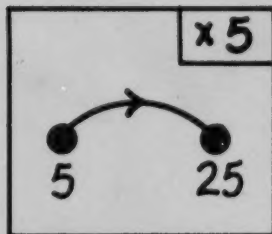
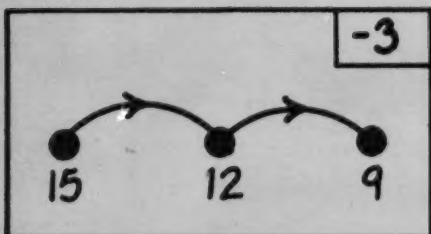
T: There are five arrow pictures and five tags on the board.
Each tag is for one of these arrow pictures.

Point to Figure 1.

T: Which tag is for this arrow picture?

Have some students whisper the answer in your ear and have one of them write "-3" in blue on the picture and label the midpoint (12). Follow a similar procedure for the four other pictures.

Answers:



Exercise 2:

Draw on the board:

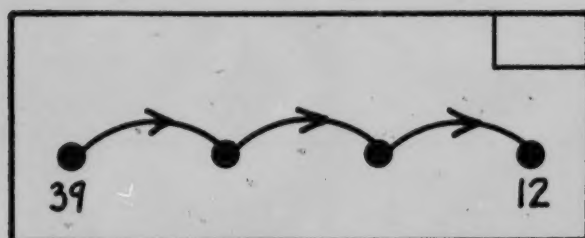
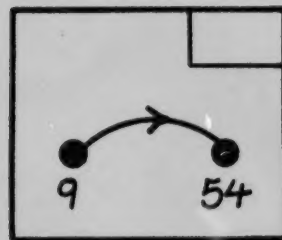
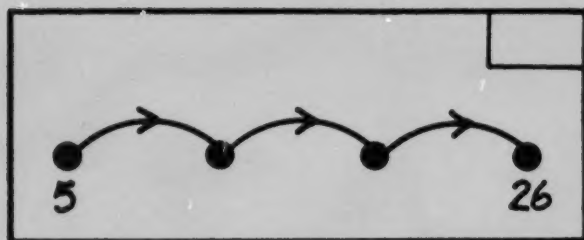
$\boxed{\times 3}$

$\boxed{+7}$

$\boxed{-9}$

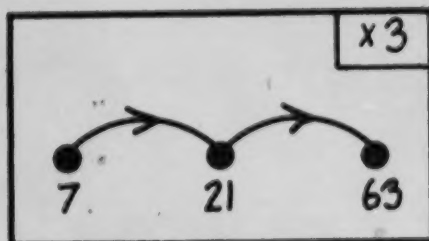
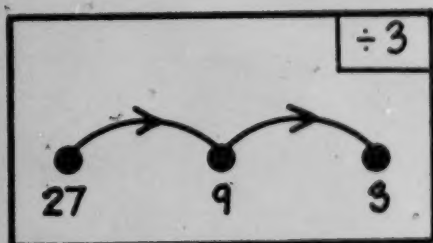
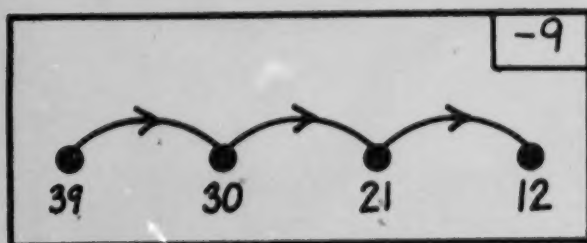
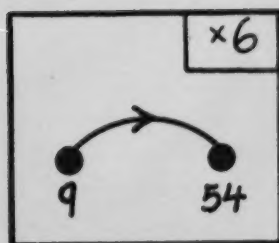
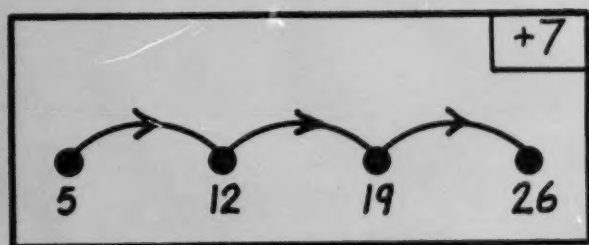
$\boxed{\div 3}$

$\boxed{\times 6}$



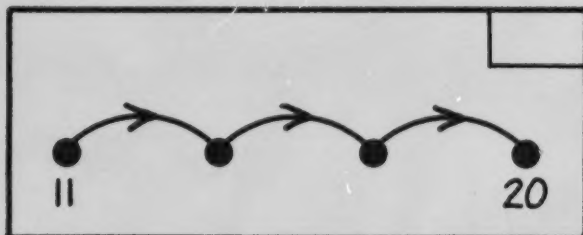
Follow the same procedure as for Exercise 1.

Answers:



Exercise 3:

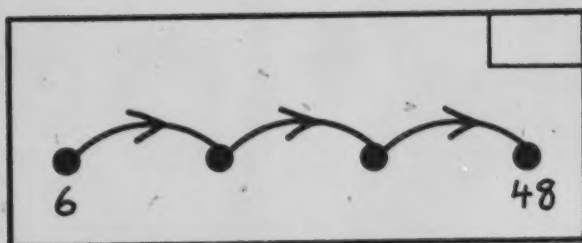
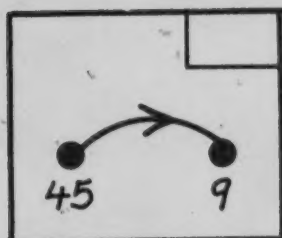
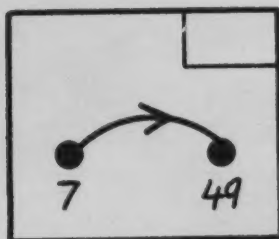
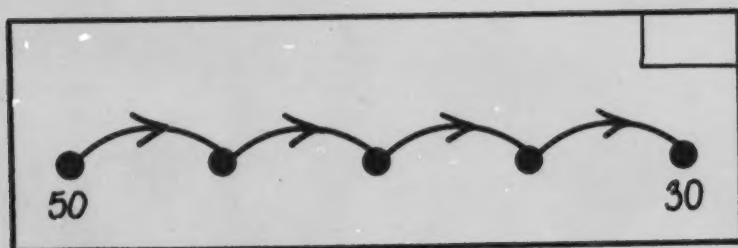
Draw this picture on the board and ask the students to copy it on a sheet of paper:



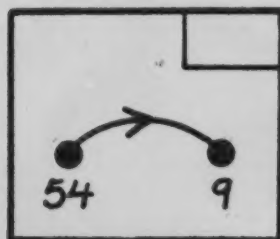
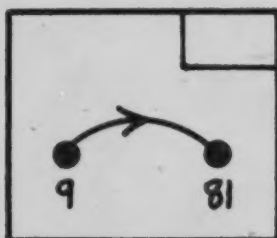
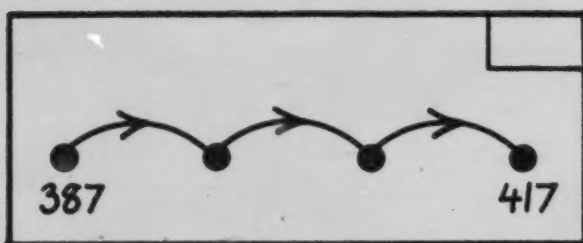
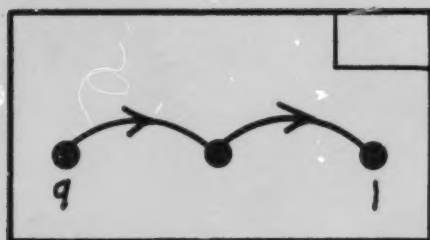
T: What could these blue arrows be for? Write the answer (in blue) in the little box in the upper right-hand corner of the picture and label the dots.

After a while, have a student explain that the blue arrows could be for $+3$. In this case, the two other dots would be for 14 and 17.

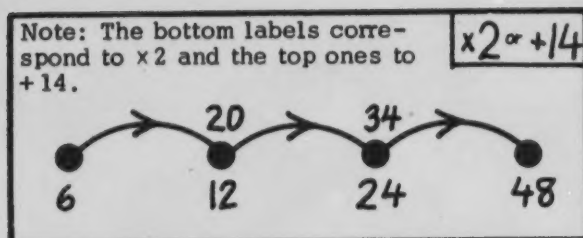
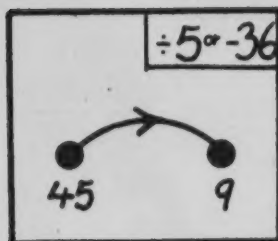
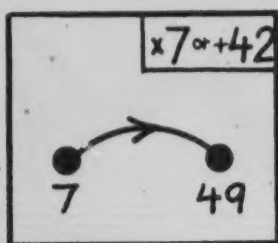
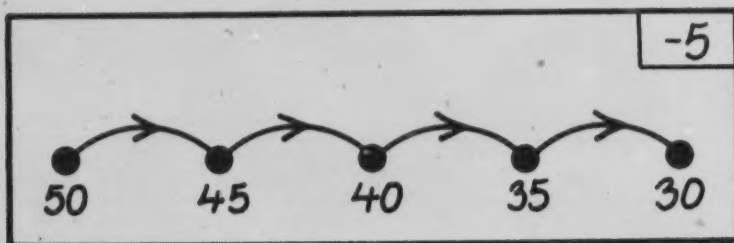
Draw the following four pictures on the board and ask the students to proceed in a similar fashion as above for each of them. Tell them that it's possible to find more than one answer for some of these problems.

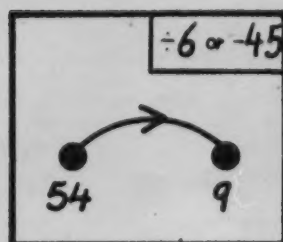
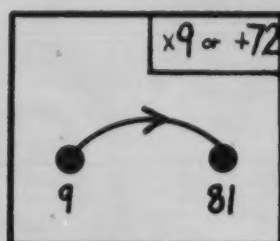
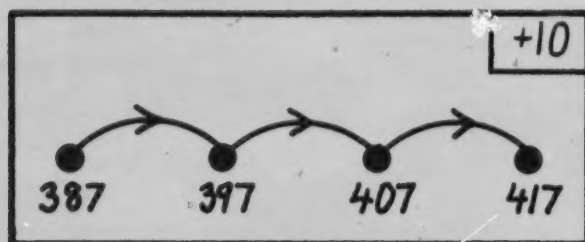
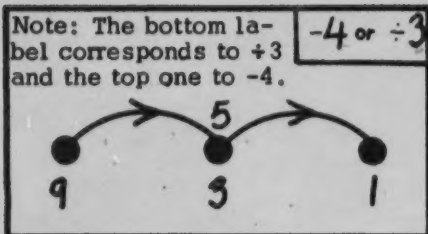


If some of your students quickly solve these four problems (and only in this eventuality), draw the following four additional ones on the board:



After a while have some students write their answers (in blue) on the board:





Worksheets N16, N17 and N18 are available for the rest of the lesson.

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Name _____

N16 *

For each arrow picture, fill in the box in the upper right-hand corner with one of these tags. Then label the dots.

$+5$ $\times 2$ -4 $\times 5$

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Name _____

N17 **

For each arrow picture, fill in the box in the upper right-hand corner with one of these tags. Then label the dots.

-7 $+20$ $\times 7$ $\div 4$ $\times 3$

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Name _____

N18 ***

In each of these pictures fill in the box in the upper right-hand corner with a label for the arrows. Then label the dots.

N18

$+16\frac{1}{3}$ is also a correct label for the arrows here in which case the two unlabeled dots are $23\frac{1}{3}$ and $39\frac{2}{3}$.

-3.5 is also a correct label for the arrows here in which case the two unlabeled dots are 8.5 and 5.

1.2 Match the Tags #1Exercise 1:

Draw on the board:



$+12$

$\times 10$

$+5$

$\times 5$

-15

$+45$

$\div 4$

$+36$

$\div 2$

$+18$

-6

$\times 2$

T: We have to match the red tags and the blue tags.

Point to the +12 red tag.

T: If the red arrow were for +12, what number would be here? (Point to the midpoint.)

S: 32, because $20 + 12 = 32$.

T: And which tag would you choose for the blue arrow?

S: The +18 blue tag because $32 + 18 = 50$.

Draw a line between the +12 red tag and the +18 blue tag.

Point to the x10 blue tag.

T: If the blue arrow were for x10, what number would be here? (Point to the midpoint.)

S: 5, because $5 \times 10 = 50$.

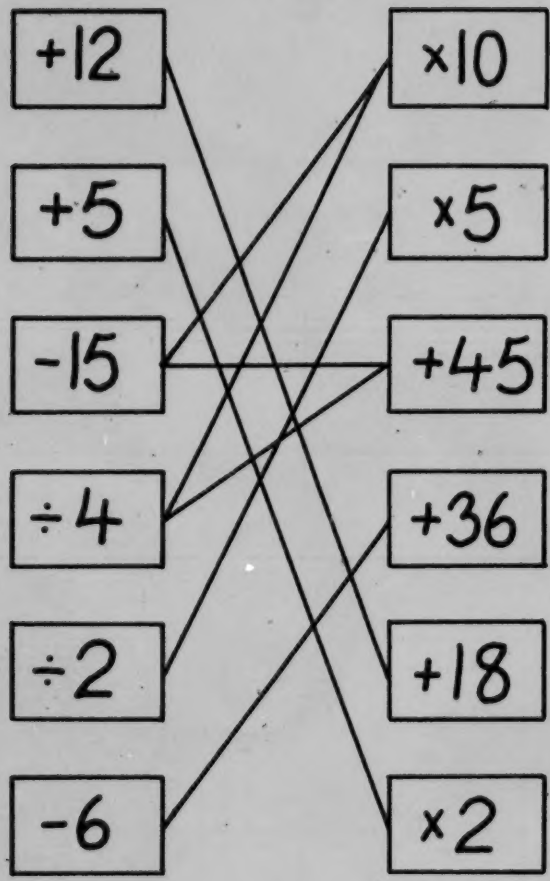
T: And which tag would you choose for the red arrow?

S: The -15 red tag because $20 - 15 = 5$.

S: You could also choose the +4 red tag because $20 \div 4 = 5$.

Draw a line between the -15 red tag and the x10 blue tag, and a line between the +4 red tag and the x10 blue tag.

Follow a similar procedure for the other tags. Alternate red and blue. When it's over, the tags should be matched in this way:



Exercise 2:

Draw this picture on the board and ask the students to copy it on a sheet of paper:



$\longrightarrow$	$\longrightarrow$
$\times 4$	
-4	
	$+10$
	-10

Point to the red arrow and ask:

T: What could the red arrow be for? You are completely free to label it in any way you wish.

Let's suppose that a student suggests +5. Write "+5" in red in the first line of the chart.

T: If the red arrow were for +5, what number would be here? (Point to the midpoint.)

S: 10, because $5 + 5 = 10$.

T: And what could the blue arrow be for?

S: +30 or $\times 4$.

Write "+30 or $\times 4$ " in blue in the first line of the chart.

Have the students work on their own for a while and try to complete the chart. If you have some fast workers, ask them to draw a second chart and to invent new solutions for the same problem. Encourage them to create solutions which introduce negative numbers (or decimal numbers) for the midpoint.

After a while, have students write answers on the board and explain them to the whole group.

For instance,



$\longrightarrow$	$\longrightarrow$
$+5$	$+30$ or $\times 4$
$+40$	-5
$\times 4$	$\times 2$ or $+20$
-4	$+39$ or $\times 40$
$+25$ or $\times 6$	$+10$
$+45$ or $\times 10$	-10
-6	$+41$
-25	$+60$

Worksheets N19 and N20 are available for the rest of the lesson.

If some of your students ask for a *** problem and have successfully solved the two worksheets, draw this picture on the board and ask them to complete the chart: (The answers for the six lines in which one of the two functions is given are written below in black. When copying the chart on the board, be sure to omit these answers.)



→	→
$\times 2$	$+20$
$+15$	$\times 2$
-20	$+120$
$+50$	-20
$\div 2$	$+95$ or $\times 20$
$+180$	$\div 2$

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Name _____

N19 *

Match:

$\times 3$	$\times 10$
$\div 2$	$\div 2$
-3	-20
$+90$	$\times 50$
$\times 7$	$+20$
-9	$+43$

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Name _____

N20 **

Complete the chart:

$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
$+30$	$\times 2$	$+30$	$\times 2$
$\div 2$	$+90$	$\times 4$	$+20$
-20	$+100$	$\times 10$	$\div 2$
$\times 2$	$+60$	$+20$	$\times 100$
$\div 5$	$+75$	-15	$+95$
$\times 6$	-20	-25	$+105$

N20

(Many other solutions are possible.)

1.3 Match the Tags #2

Exercise 1:

Draw on the board:



$\times 20$

$\times 2$

-20

$\div 2$

$\times 2$

-20

$+20$

$+25$

$+30$

$+15$

$+120$

$+50$

$+95$

Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: We have to match the red tags and the blue tags.

Point to the $\times 2$ red tag.

T: If the red arrows were for $\times 2$, what numbers would be here (point to a) and here (point to b)?

S: 40 (for a) and 80 (for b).

Write "40" and "80" on the picture.

T: And which tag would you choose for the blue arrow? (Have some students whisper the answer in your ear and then call on someone to give it out loud.)

S: +20.

Draw a line between the $\times 2$ red tag and the +20 blue tag.

Erase "40" and "80" and point to the +2 red tag.

T: If the red arrows were for +2, what numbers would be here (point to a) and here (point to b)?

S: 10 (for a) and 5 (for b).

Write "10" and "5" on the picture.

T: And which tag would you choose for the blue arrow? (Have some students whisper the answers in your ear and then call on someone to give it out loud.)

S: +95.

S: $\times 20$.

Then draw a line from the +2 red tag to the x20 blue tag and another line from the +2 red tag to the +95 blue tag.

Erase "5" and "10" and point to the -20 blue tag.

T: If the blue arrow were for -20, what number would be here (point to b)?

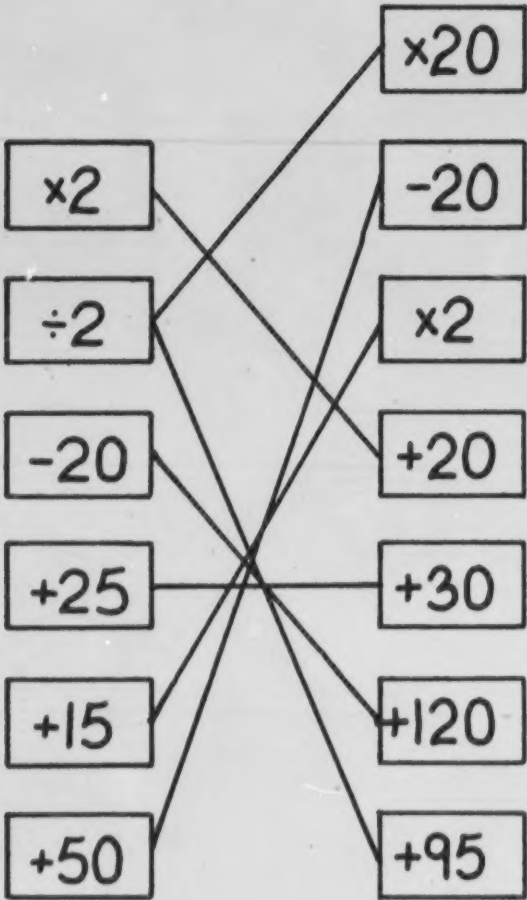
S: 120.

Write "120" on the board.

T: And which tag would you choose for the red arrows? (Answer: +50)

Have some students whisper the answer in your ear and have one of them explain why the red arrows should be for +50. Draw a line from the +50 red tag to the -20 blue tag. Erase "120" and follow similar steps for the other tags. Alternate red and blue for the first tag to which you point.

When this exercise is completed, your board should look like this:



Exercise 2:

Draw this picture on the board and ask the students to copy it on a sheet of paper:



$\longrightarrow$	$\longrightarrow$
$\times 2$	
$+10$	
	$+10$
	-22
$\div 2$	
-2	

Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

Point to the red arrows and ask:

T: What could the red arrows be for? You are completely free to label them in any way you wish.

Let's suppose that a student suggests +2. Write "+2" in red in the first line and first column of the chart.

T: If the red arrows were for +2, what numbers would be here (point to a) and here (point to b)?

S: 4 (for a) and 6 (for b).

Write "4" and "6" on the picture.

T: And what could the blue arrow be for? (Ask for whispers in your ear and then call on someone to give the answer out loud.)

S: +14.

Write "+14" in blue in the first line and second column of the chart.

Have the students work on their own for a while and try to complete the chart. If you have some fast workers, ask them to invent more solutions for the same problem. Encourage them to create solutions that introduce negative numbers (or decimal numbers) for the numbers a and b.

After a while, have students write answers on the board and explain them to the whole group.

One possible solution follows:



$\longrightarrow$	$\longrightarrow$
$+2$	$+14$
$\times 3$	$+2$
$\times 5$	-30
$\times 2$	$+12$
$+10$	-2
$+4$	$+10$
$+20$	-22
$\div 2$	$+19.5$
-2	$+22$
$+19$	$\div 2$
$+50$	-82
-10	$+38$

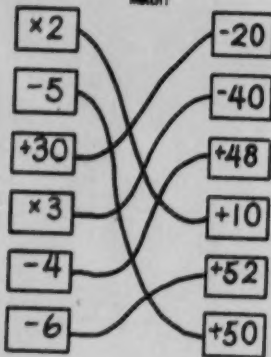
Worksheets N21, N22 and N23 are available for the rest of the lesson.

Name _____

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N21 *



Match:



Name _____

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N22 **



Complete the chart:

→	→
+20	+20
-20	+100
÷ 2	+90 x10
+5	+50
+10	+40
-10	+80

→	→
-5	+70
-10	x5
-30	+120
-40	+140
+80	÷ 2
-15	x10

Name _____

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N23 ***



Complete the chart:

→	→
x2	+40
x20	-50
x40 +390	÷ 2
+15	x2
x3	+35
x30	-100

→	→
-30	+60
-20	+55
-9 ÷10	x10
+140 x14	-25
+40	+25
x5	+25

N22

(Many other solutions are possible.)

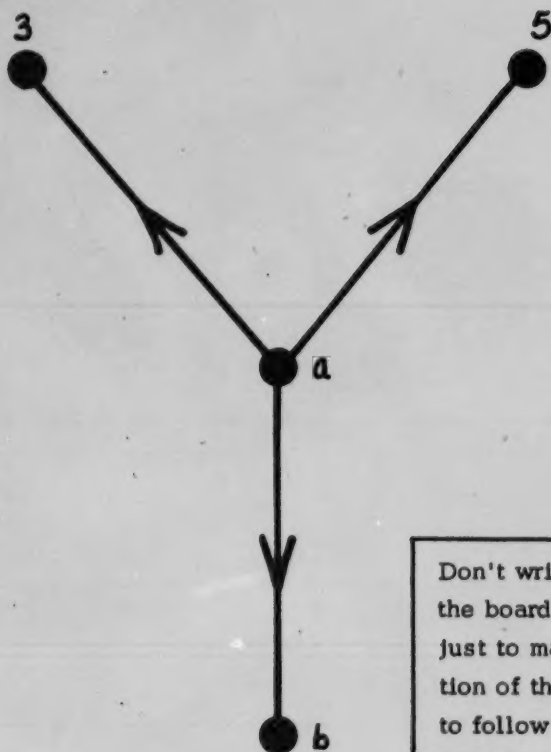
N23

(Many other solutions are possible.)

1.4 Find the Missing Arrows

Exercise 1:

Draw on the board:



Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: What could these red arrows be for? Many answers are possible.

Let's suppose that a student suggests +1.

T: If the red arrows were for +1, what number would be here (point to a)?

The student will probably recognize that the arrows cannot be for +1.

T: If you look at this arrow (point to the arrow from a to 5), you will have to conclude that this number (point to a) is 4.

But if you look at this arrow (point to the arrow from a to 3), then you have to conclude that the same number (point to a) is 2.

Is it possible?

S: No, the red arrows cannot be for +1.

Follow similar steps if the students suggest that the red arrows could be for -1 or $\times 2$ or $\div 2$ or other examples of those special relations such that from each dot at most one arrow of the relation can start.

In one of our fourth grade development classes, one student said:

S: The red arrows could be for +1 or -1.

T: If the red arrows were for +1 or -1, what number would be here (point to a)?

S: 4.

T: It's correct. But what number would be there (point to b)?

The student recognized the difficulty because the only possible answers are 3 or 5 and we have accepted the convention that two different dots represent two different numbers. Therefore the red arrows cannot be for +1 or -1. If no child suggests it, write on the board:

I am larger than you

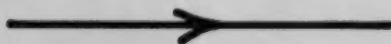


T: If the red arrows were for "I am larger than you" what number could be here (point to a)? (Answer: 10; 100; $5\frac{1}{2}$; 5.1; and so on. Any number larger than 5.)

Label the dot at a with one of the numbers suggested by a student and ask a similar question for b. (Answer: b can be any number smaller than a and different from 3 and 5.)

Erase "I am larger than you" from the board and write:

I am at least 10 smaller than you



T: If the red arrows were for "I am at least 10 smaller than you", what number could be here (point to a)? (Answer: -7 ; -10 ; -50 ; $-7\frac{1}{2}$; -7.2 ; and so on. Any number equal to or smaller than -7 .)

Label the dot at a with one of the numbers suggested by a student, and ask a similar question for b. (Answer: b has to be at least 10 larger than a and different from 3 and 5.)

Erase "I am at least 10 smaller than you" from the board and write:

I am a multiple of you



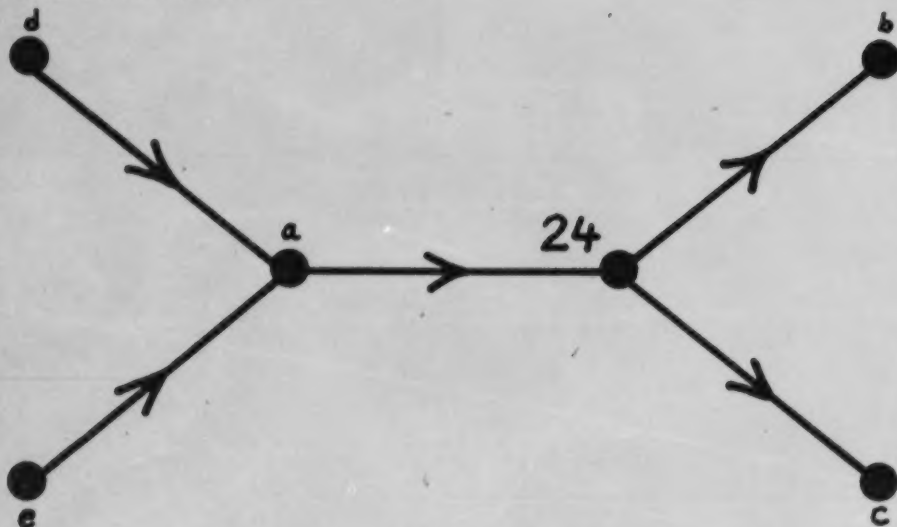
T: If the red arrows were for "I am a multiple of you", what number could be here (point to a)? Write your answer on a sheet of paper. (Answer: 0; 15; 30; 45; 60; 75; and so on. Any multiple of 15.)

Label the dot at a with one of the numbers suggested by the students, for instance 30.

T: What number could be here (point to b)? Write your answer on a sheet of paper. (Answer: 1; 2; 6; 10; 15. Any divisor of 30 different from 3, 5 and 30.)

Exercise 2:

Draw on the board:

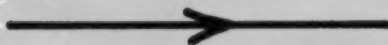


Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

- T: What could these red arrows be for?
 What could these blue arrows be for?
 Many answers are possible.

This is an open question. You will have to adjust your lesson according to the suggestions of your pupils. Here, we will describe how to conduct the lesson if the students suggest:

I am a multiple of you



I am at least 20 larger than you



T: If the red arrows were for "I am a multiple of you", what numbers could be here (point to b and c). Write your answer on a sheet of paper. (Answer: 12 or 8 or 6 or 4 or 3 or 2 or 1.)

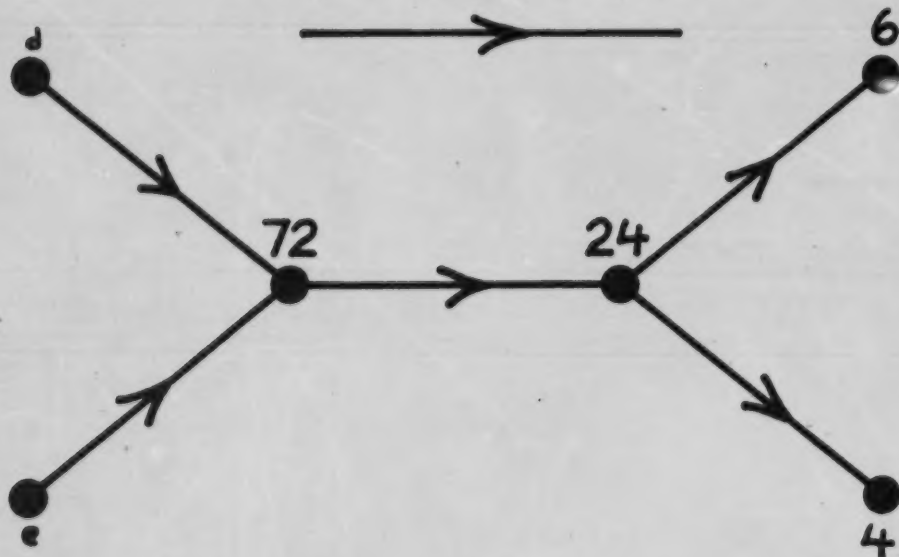
What numbers could be here (point to a)? (Answer: 48; 72; 96; 120; and so on. Any multiple of 24 with the exception of 24.)

Write on the picture three numbers suggested by your students for a, b and c, for instance,

I am a multiple of you

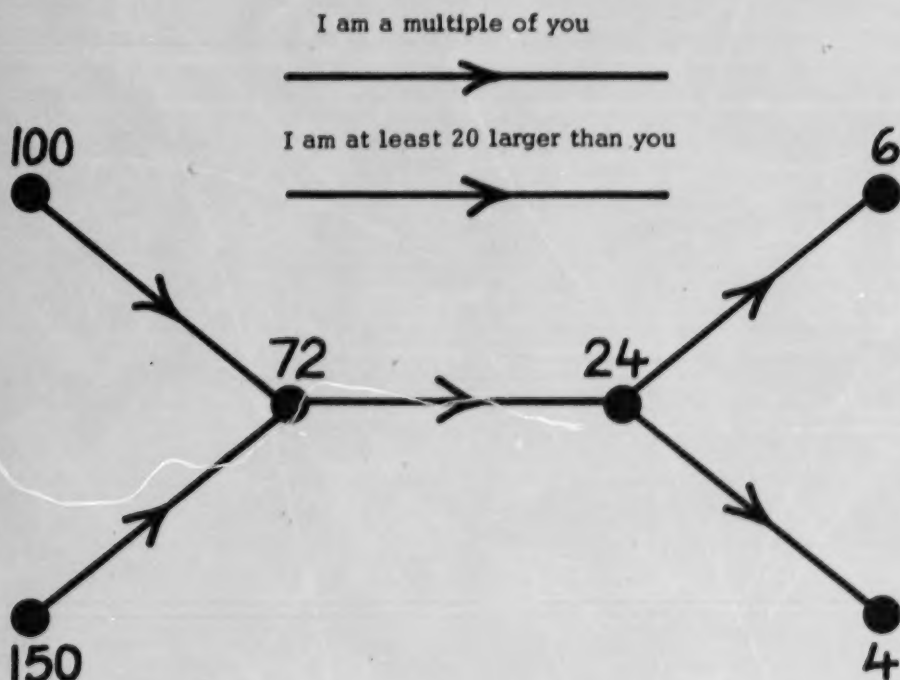


I am at least 20 larger than you



T: What numbers could be here (point to d and e)? (Answer: Any number equal to or larger than 92.)

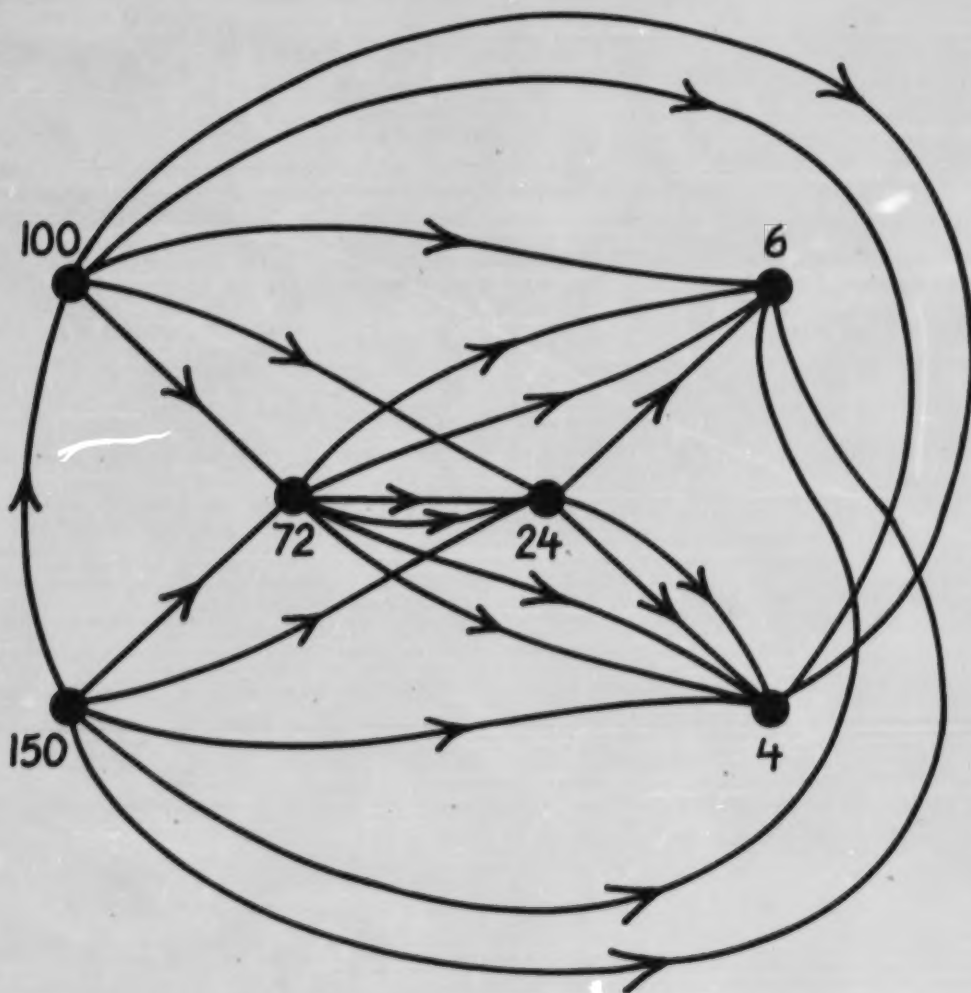
Label the dots at d and e with two numbers suggested by your students, for instance,



T: In this picture, we could draw some more red arrows and some more blue arrows.

First have students trace red arrows on the board. Then draw them yourself: from 72 to 6; from 72 to 4; from 100 to 4 and from 150 to 6. Remind the students that there should be red loops at every dot but that you're not going to draw them because the picture would become too cluttered.

Afterwards, have the students trace all the missing blue arrows. Draw them yourself.



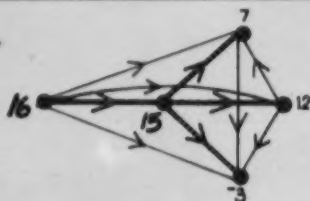
Worksheets N24, N25, N26 and N27 are available for the rest of the lesson.

Name _____

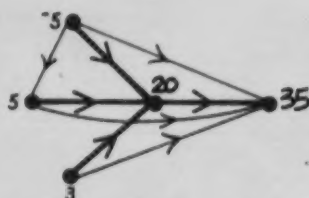
N24 *

In both pictures, label the dots. Many answers are possible. Then draw all the missing arrows.

I am larger than you



I am at least 10 smaller than you



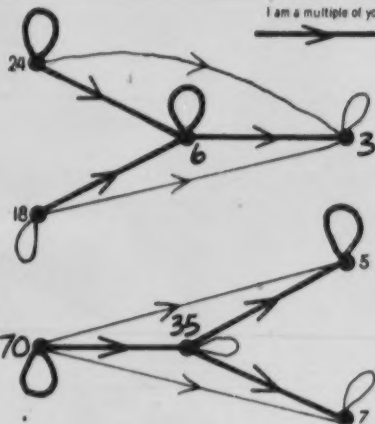
N24, N25, N26, N27: (Many other solutions are possible.)

Name _____

N25 ***

Label the dots. Many answers are possible. Then draw all the missing arrows.

I am a multiple of you



Name _____

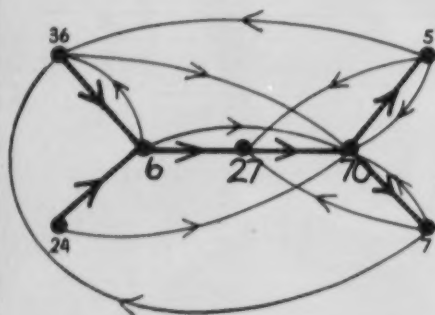
N26 **

Label the dots. Many answers are possible. Then draw all the missing arrows. (It's not necessary to draw the red loops as it would make the picture too crowded.)

I am a multiple of you



I am at least 20 smaller than you



Name _____

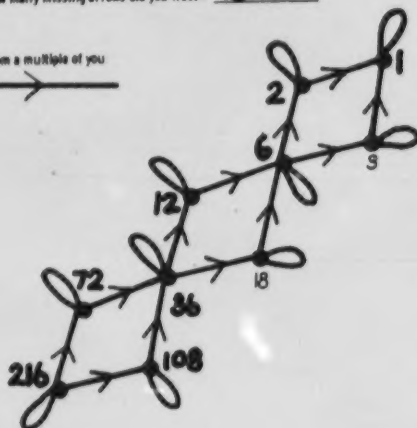
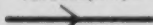
N27 ***

Label the dots. Many answers are possible. Trace with your finger all the missing arrows but don't draw them.

How many missing arrows did you trace?

30

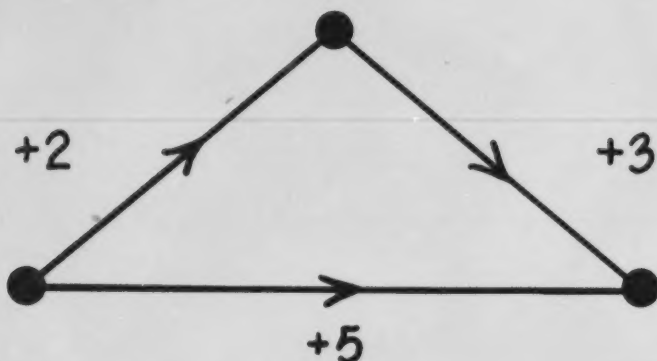
I am a multiple of you



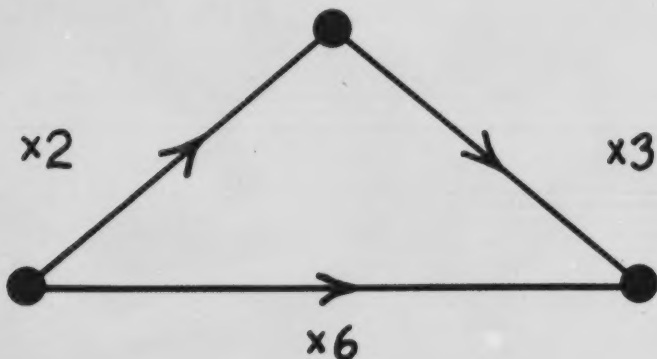
2

COMPOSITION OF FUNCTIONS

This chapter deals with what happens when you apply a sequence of functions one after another. This is known as "composing" the functions concerned in the specified order. As it happens, it's a fairly tricky business, and one is likely to stumble into all kinds of unexpected pitfalls unless the subject is developed very carefully. For example, it is clear that, if you add 2 to a number and then you immediately add 3 to the result, the net effect is that of adding 5 to the original number.



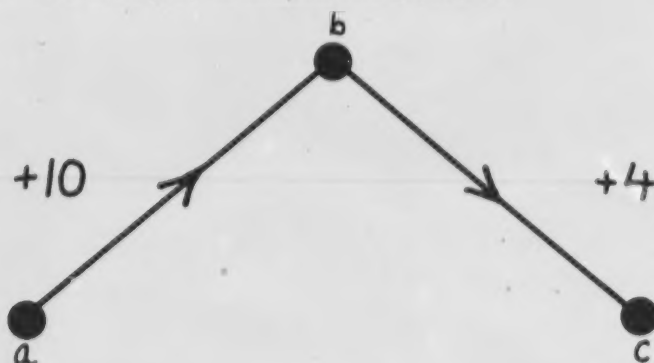
What could be more natural than to suppose that if you multiply a number by 2 and then immediately multiply the result by 3, the net effect is that of multiplying the original number by 5? This is not so. The true result is shown in the following arrow diagram:



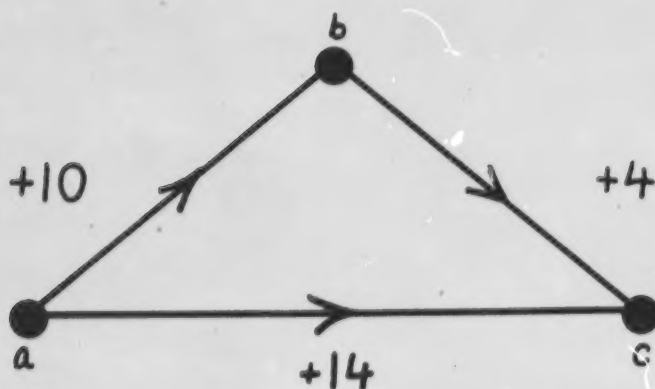
To avoid unnecessary complications, a restriction is made in the lessons of this chapter so that you don't mix addition and multiplication. This means that composites of functions involving "+" and "-" may be considered; so too can composites of functions involving "x" and "÷"; but for these lessons we do not consider the composition of one function involving "+" or "-" with another involving "x" or "÷".

The teaching strategies involved run along two lines:

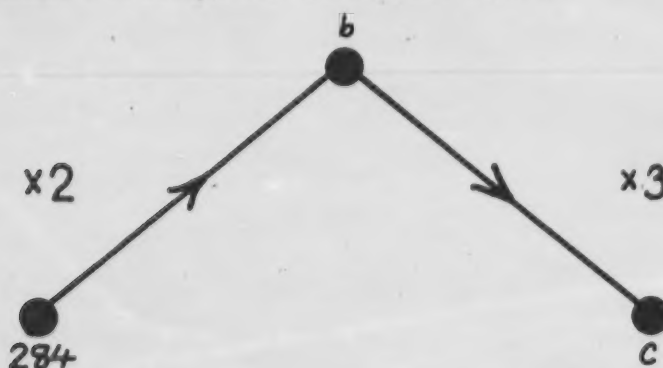
- a) When dealing with functions involving addition and/or subtraction, concentrate on the relative sizes of the numbers represented by the dots. For example, in the following picture:



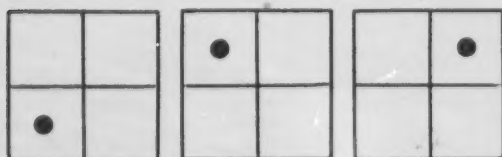
looking at the red arrow, we see that b is ten more than a. Then, we see from the blue arrow that c is four more than b. So c is four more than a number which is ten more than a. That means c must be fourteen more than a. (Replace "a" by some specific numbers and you will find this to be correct.) Hence we can always draw the following green arrow, no matter what numbers are represented by the three dots:



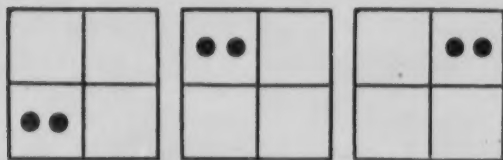
- b) When dealing with functions involving multiplication and/or division, the Minicomputer is brought into play. For an example, consider Exercise 2 of Section 2.1, paraphrased below:



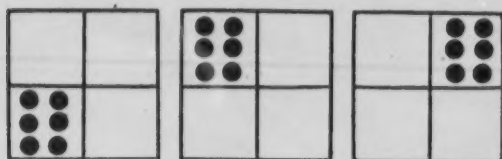
A student is asked to put 284 on the Minicomputer:



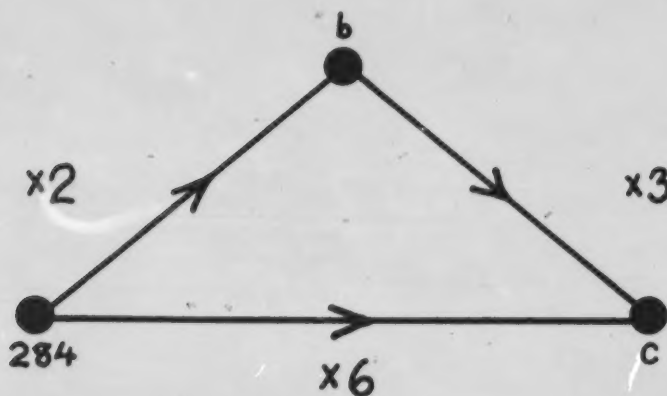
The teacher points to b and asks a student to put this number on the Minicomputer:



Next, the teacher points to c and emphasizes the fact that this number is 3 times the number that is now on the Minicomputer. A student is then asked to put this number on the Minicomputer:



The students conclude that this number is 6 times 284 and that the green arrow could be for $\times 6$.



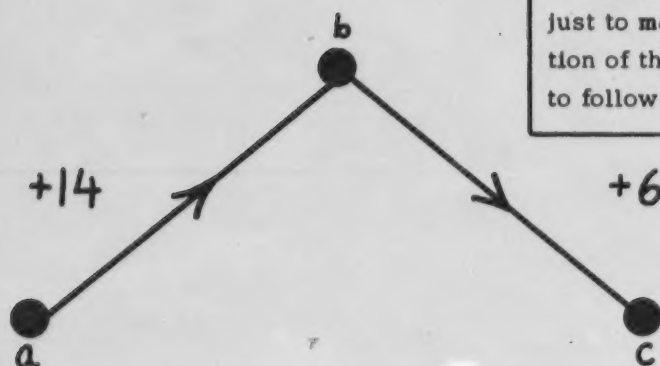
From there, it is but a small step until it is appreciated that, if the green arrow is for $\times 6$, then all possible situations will have been taken care of, not just the example that was considered.

CAPSULE LESSON SUMMARIES

- Section 2.1:** Using the Minicomputer and arrow diagrams, investigate the composition of certain numerical functions; e.g., $+14$ followed by $+6$ and $\times 2$ followed by $\times 3$. (Worksheets N28, N29, N30)
- Section 2.2:** Investigate commutativity of the composition of certain numerical functions; e.g., $+2$ followed by $+10$ and $\times 2$ followed by $\times 10$. Contrast the result of composing $+10$ with itself (i.e., $+10$ followed by $+10$) with that of $\times 10$ followed by $\times 10$ (using the Minicomputer). (Worksheets N31, N32, N33)
- Section 2.3:** Using long strings of numerical functions; e.g., various combinations of $\times 2$ and $\times 10$ and their converses $\div 2$ and $\div 10$ respectively, examine the composition of the two functions, $\times 20$ and its converse, $\div 20$. Also include some applications to the decimal numbers. (Worksheets N34, N35, N36)

2.1 Composition of Functions #1Exercise 1:

Draw on the board:



Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: How much larger is this number (point to b) than this one (point to a)?

S: 14 larger.

T: How much larger is this number (point to c) than this one (point to b)?

S: 6 larger.

T: What is the smallest number in this picture? Come to the board and point to it.

S: This one (pointing to a).

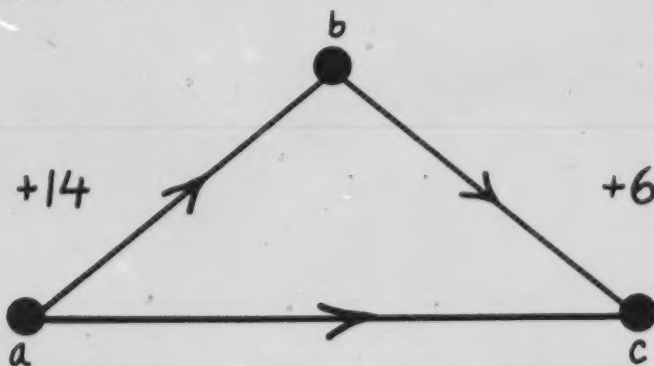
T: What is the largest number in this picture? Come to the board and point to it.

S: This one (pointing to c).

T: How much larger is this number (point to c) than this one (point to a)?

S: 20 larger.

Draw this green arrow:



T: What could this green arrow be for?

S: +20.

Write "+20" in green on the board.

T: If this dot (point to a) were for 21, what number would be here (point to b) and here (point to c)?

Have the students write the answers (35 and 41) on a sheet of paper and notice that 41 is 20 more than 21.

T: If this dot (point to b) were for 44, what number would be here (point to a) and here (point to c)?

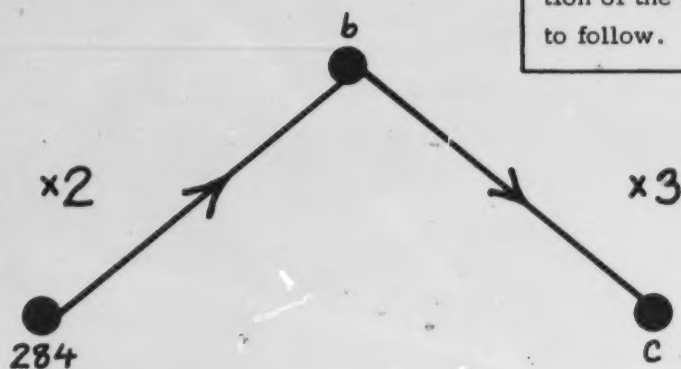
Have the students write the answers (30 and 50) on a sheet of paper and notice that 50 is 20 more than 30.

T: If this number (point to c) were 64, what number would be here (point to b) and here (point to a)?

Have the students write the answers on a sheet of paper (58 and 44) and notice that 64 is 20 more than 44.

Exercise 2:

Display three Minicomputer boards and draw on the board:

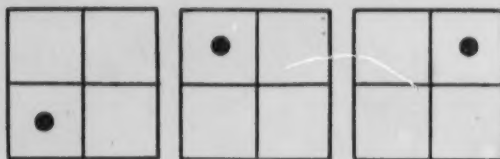


Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

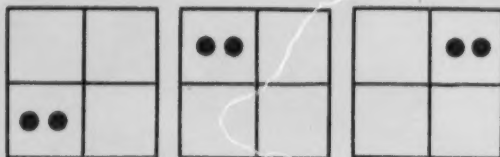
44

256

Ask a student to put 284 on the Minicomputer:

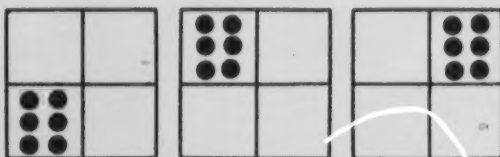


Point to b and ask a student to put this number on the Minicomputer:



Point to c; emphasize the fact that this number is 3 times the number that is now on the Minicomputer.

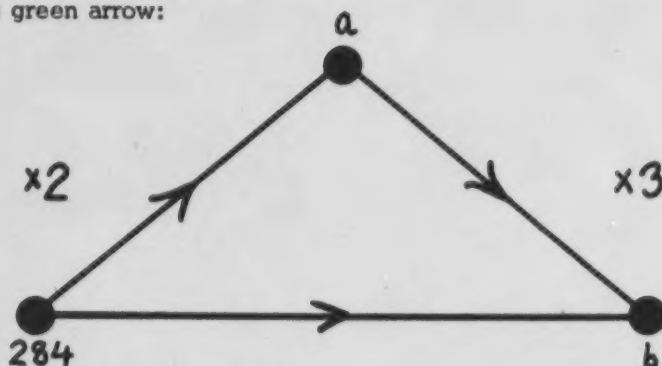
Ask a student to put this number (c) on the Minicomputer:



T: How many times 284 is this number (point to c)?

S: Six times.

Draw this green arrow:



T: What could this green arrow be for?

S: $\times 6$.

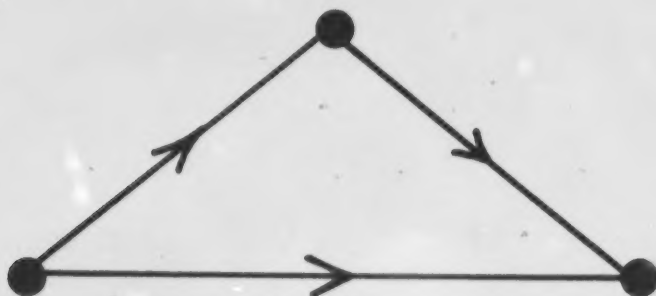
Write " $\times 6$ " in green on the board.

Erase "284" and write "4" instead. Ask the students to calculate the two other numbers ($8 = 2 \times 4$; $3 \times 8 = 24$). Emphasize the fact that $24 = 6 \times 4$.

Erase "4" and do similar examples successively with 5; 7; 8; 10; 12 or any other number you wish.

Exercise 3:

Draw this picture on the board and give worksheet N28 to each student.



T: In each picture you are to label the arrows and the dots in many different ways. You are completely free, with one exception:

In the same picture, you are not to mix plus (+) or minus (-) arrows with times (x) or divided by ($\div$) arrows.

After a short period of time, have two or three students draw one of their pictures on the board. Have the other students check to see if the pictures are correct and then let them continue working on their own pictures.

For the students who successfully complete the six arrow pictures, worksheets N29 and N30 are available. (You may need to remind the students that if in a given picture two arrows are of the same color, then they both are for the same function in that problem.)

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Name _____

N 28 *

$+10$ -6 $+4$	$\times 2$ $\times 3$ $\times 6$
$+5$ $+15$ $+20$	$\times 2$ $\times 10$ $\times 20$
-45 $+100$ $+55$	$\times 5$ $\times 20$ $\times 100$

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Name _____

N 29 *

Label the dots and complete the tag for each green arrow with one of the following tags:

$\times 6$ $\times 9$ $+10$ $+7$ $\times 8$ $+6$

$+3$ 15 12 22 10	$\times 3$ 15 5 30 6
$+3$ 42 39 45 6	$\times 3$ 12 4 36 9
$+10$ 30 20 27 7	$\times 2$ 12 6 48 8

N28

(Many other solutions are possible.)

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Name _____

N 30 **

Label the dots and complete the tag for each green arrow with one of the following tags:

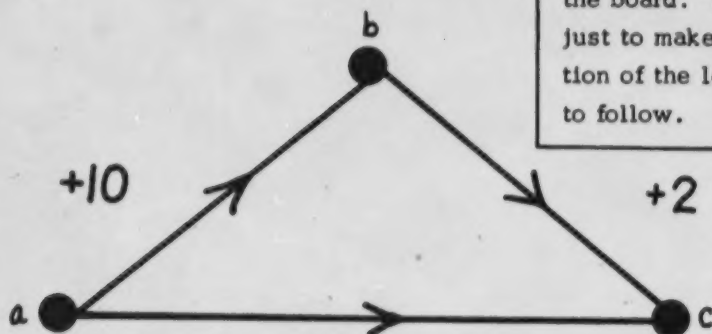
$\times 12$ $\times 25$ -25 $\times 10$ $+25$ $+12$

$\times 5$ 45 9 90 10	$+20$ 200 180 192 12
$\times 3$ 90 30 360 12	$\times 5$ 200 40 $1,000$ 25
$+40$ 415 375 400 25	-30 245 275 250 25

2.2 Composition of Functions #2

Exercise 1:

Draw on the board:



Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: What is the smallest number in this picture?

Come to the board and point to it.

S: This one (pointing to a).

T: What is the largest number in this picture?

Come to the board and point to it.

S: Here (pointing to c).

T: How much larger is this number (point to c) than this one (point to a)?

S: 12 larger.

T: What could the green arrow be for?

S: +12.

Write "+12" in green on the board.

T: If this number (point to b) were 59, what number would be here (point to a)?

S: 49.

T: What number would be here (point to c)?

S: 61.

T: Is it true that $49 + 12 = 61$?

S: Yes.

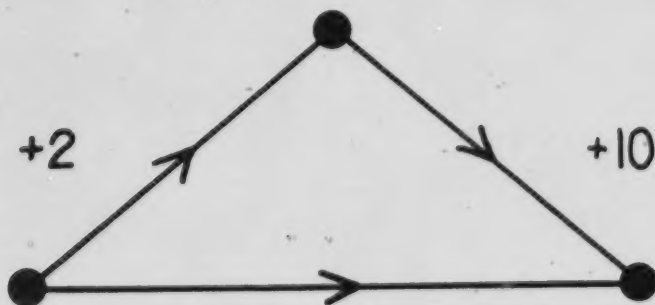
T: If this dot (point to c) were 1,000, what number would be here (point to b) and here (point to a)?

S: 998 (for b) and 988 (for c).

Have the students check that $988 + 12 = 1,000$.

Ask a similar question for one or two more numbers if you wish.

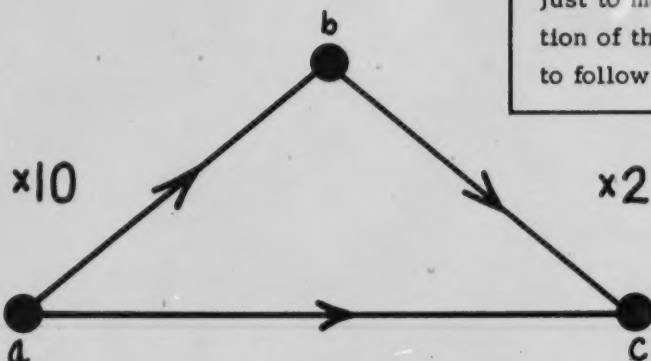
Erase "+12" and exchange the red tag and the blue tag.



Have a student explain that the green arrow could still be for +12.

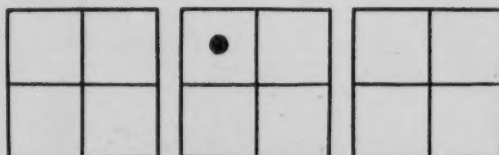
Exercise 2:

Draw on the board:



Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

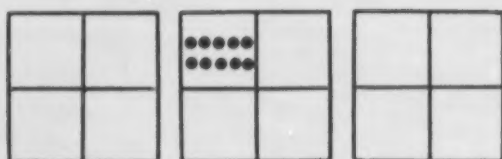
Display three Minicomputer boards and put one checker on the 80-square.



T: If this number (point to a) were 80, what number would be here (point to b)?

S: 800 because $10 \times 80 = 800$.

Put 10 checkers on the 80-square:



T: What number would be here (point to c)?

S: 1,600 because $2 \times 800 = 1,600$.

T: How many checkers are on the 80-square now?

S: 10.

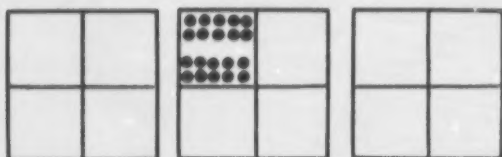
T: How many checkers altogether do we need on the 80-square in order to get 1,600 which is the number here (point to c)?

S: 20. We have to double 10.

T: So how many checkers do we still have to put on the 80-square?

S: 10 more.

Put the additional ten checkers on the Minicomputer.



T: This number (point to c) is how many times this number (point to a)?

S: 20 times.

T: What could the green arrow be for?

S: $\times 20$.

Write " $\times 20$ " in green on the picture.

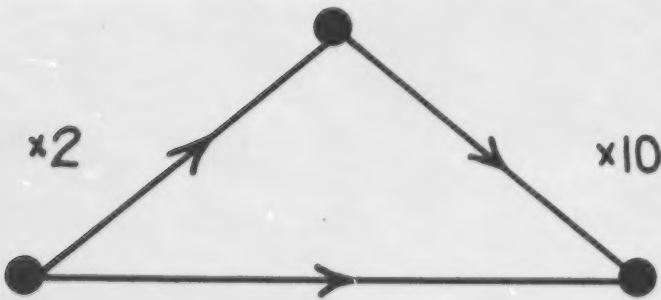
T: If this number (point to c) were 1,000, what number would be here (point to b) and here (point to a)?

S: 500 (for b) and 50 (for a).

Have the students check that $20 \times 50 = 1,000$.

Ask a similar question for two or three more numbers if you wish.

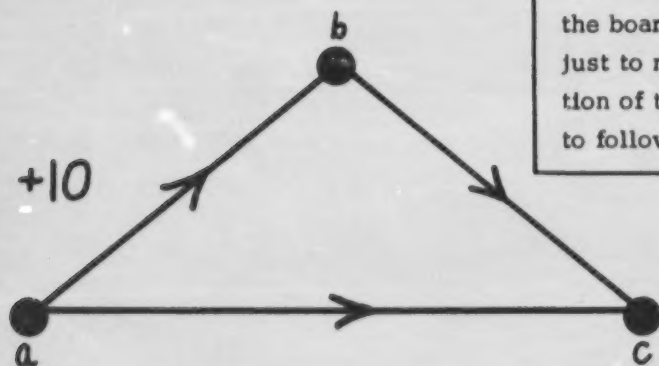
Erase " $\times 20$ " in green and exchange the red tag and the blue tag.



Have a student explain why the green arrow could still be for $\times 20$.

Exercise 3:

Draw on the board:



Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: What is the smallest number in this picture?

Come to the board and point to it.

S: Here (pointing to a).

T: What is the largest number in this picture?

Come to the board and point to it.

S: This one (pointing to c).

T: How much larger is this number (point to c) than this number (point to a)?

S: 20 larger.

T: What could this green arrow be for?

S: +20.

Write "+20" in green on the board.

T: If this number (point to c) were 14, what number would be here (point to b) and here (point to a)?

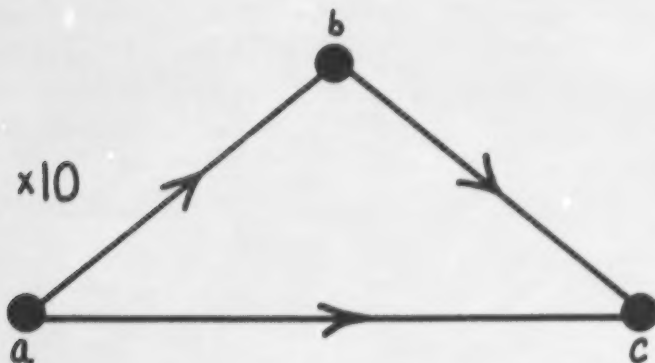
S: 4 (for b) and -6 (for a).

Have the students check that $-6 + 20 = 14$.

Ask a similar question for some other numbers, if you wish.

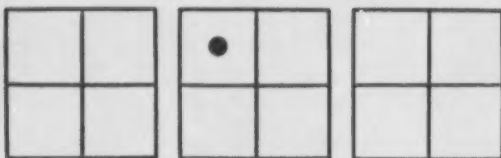
Exercise 4:

Draw on the board:



Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

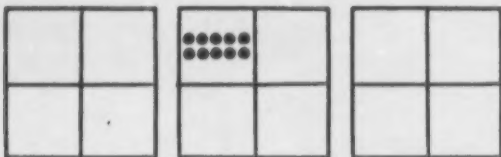
Display three Minicomputer boards and put one checker on the 80-square.



T: If this number were 80 (point to a), what number would be here (point to b)?

S: 80 because $10 \times 80 = 800$.

Put 10 checkers on the 80-square.



T: What number would be here (point to c)?

S: 8,000 because $10 \times 800 = 8,000$.

T: How many checkers should be on the 80-square altogether in order to have 8,000 on the Minicomputer?

S: 100 checkers.

T: How many are missing?

S: 90 checkers.

T: This number (point to c) is how many times this number (point to a)?

S: 100 times because $8,000 = 100 \times 80$ (or because $10 \times 10 = 100$).

T: What could this green arrow be for?

S: $\times 100$.

Write " $\times 100$ " in green on the board.

T: If this number (point to b) were 750, what number would be here (point to a) and here (point to c)?

S: 75 (for a) and 7,500 (for c).

Have the students notice that $7,500 = 100 \times 75$.

T: If this number (point to b) were 5, what number would be here (point to a) and here (point to c).

S: 0.5 or 0.50 (for a) and 50 (for c).

Have the students check that $50 = 100 \times 0.50$.

Ask a similar question for some more numbers if you wish.

Worksheets N31, N32 and N33 are available for the rest of the period.

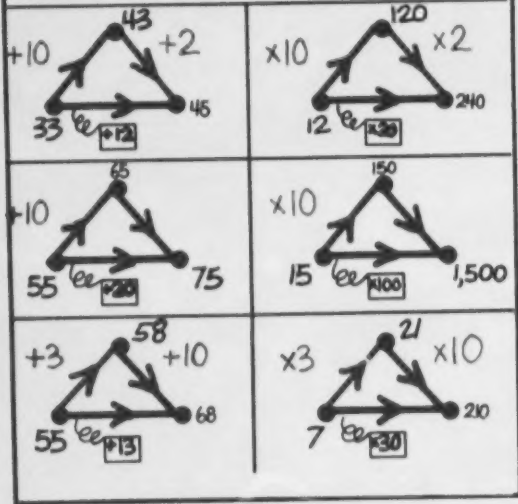
(Remind the students that in a given picture all arrows of the same color are for the same function.)

N 31 *

Name _____

Label the dots and complete the tag for each green arrow with one of the following tags:

$\times 100$ $\times 20$ $+13$ $+20$ $+12$ $\times 30$

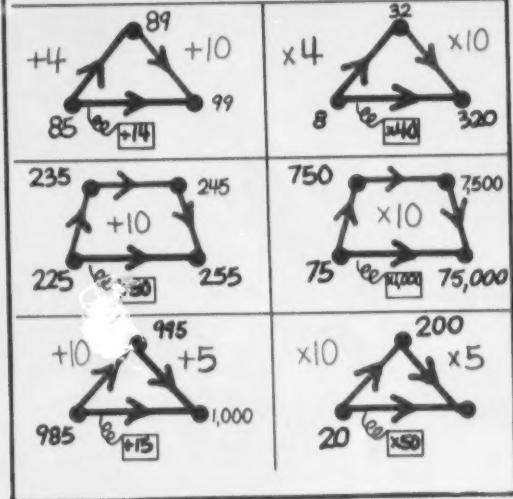


N 32 **

Name _____

Label the dots and complete the tags for each green arrow with one of the following tags:

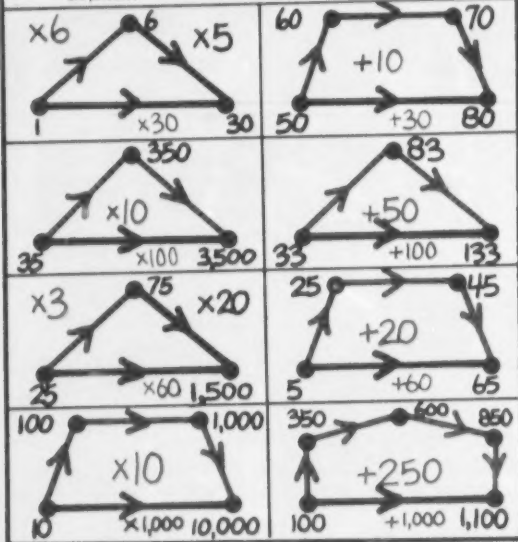
$\times 40$ $+15$ $\times 1,000$ $+14$ $\times 50$ $+30$



N 33 ***

Name _____

Label the blue arrows, the green arrows and the dots. Many solutions are possible.



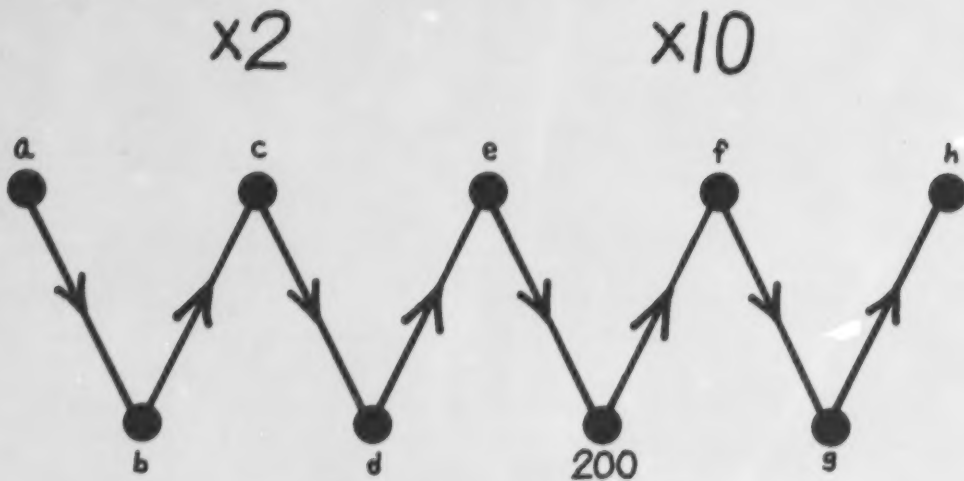
N 33

(Many other solutions are possible.)

2.3 Composition of Functions #3

Exercise 1:

Draw on the board:



Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

Point to f.

T: What is this number?

S: 400 because $2 \times 200 = 400$.

Ask a similar question successively for g, h, e, d and c. Write the numbers on the picture.

It's likely that many students will experience some difficulty in finding b and a. You could help them in the following manner:



Point to b and ask the students to write this number on a sheet of paper. Most of them will probably write " $\frac{1}{2}$ ". Ask also for the decimal writing of $\frac{1}{2}$. Emphasize the fact that

$$\frac{1}{2} = 0.5 = 0.50$$

Use the support of money notation to make this point clear.

Point to a.

T: What is this number?

Tracing the red arrow from a to b, emphasize that $10 \times$ this number equals 0.50. Write on the board:

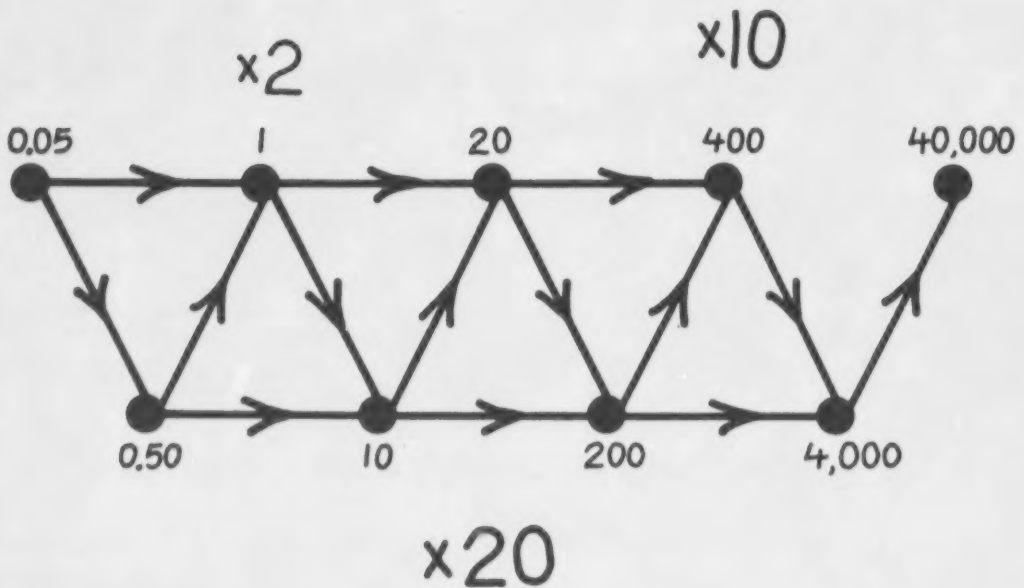
$$10 \times \boxed{} = 0.50$$

Use the support of money to make the answer clear (10 nickels is the same as 50¢ or 5 dimes):

$$10 \times \boxed{0.05} = 0.50$$

Write "0.50" and "0.05" on the picture.

Tell the students to complete the picture by using green to draw as many $\times 20$ arrows as they can.



It's likely that some students will trace a green arrow from 400 to 40,000. In that case draw a yellow arrow from 400 to 40,000 and ask:

T: How could we label this yellow arrow?

S: $\times 100$.

Write on the board:

$$4,000 \div 20 = \qquad 10 \div 20 =$$

T: The arrow picture can help you to complete these calculations. Do it on your sheet of paper.

After a while, trace (with your finger) the return arrow from 4,000 to 200.

T: How could we label this return arrow?

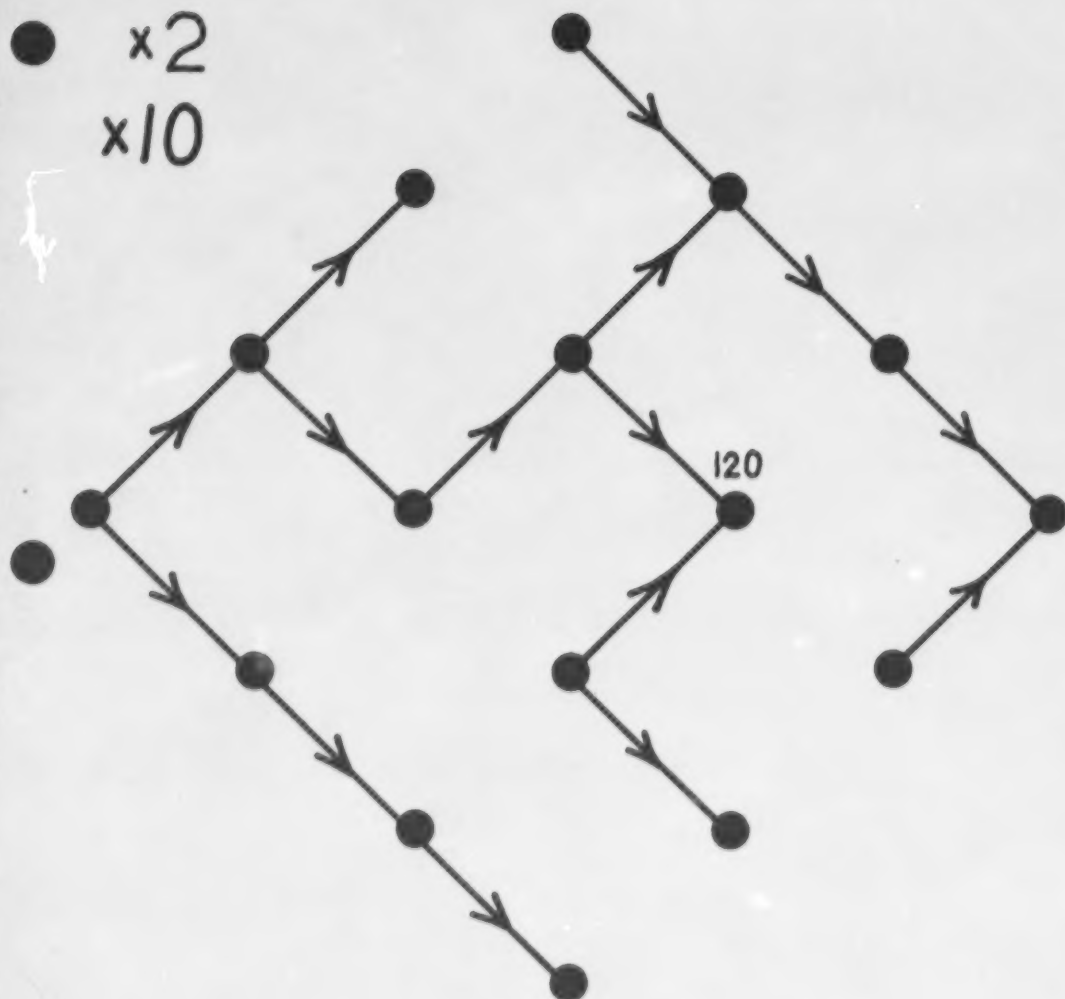
S: $+20$ or $\times \frac{1}{20}$.

Therefore the arrow picture clearly shows that $4,000 + 20 = 200$. In a similar way, it also shows that $10 + 20 = 0.50$.

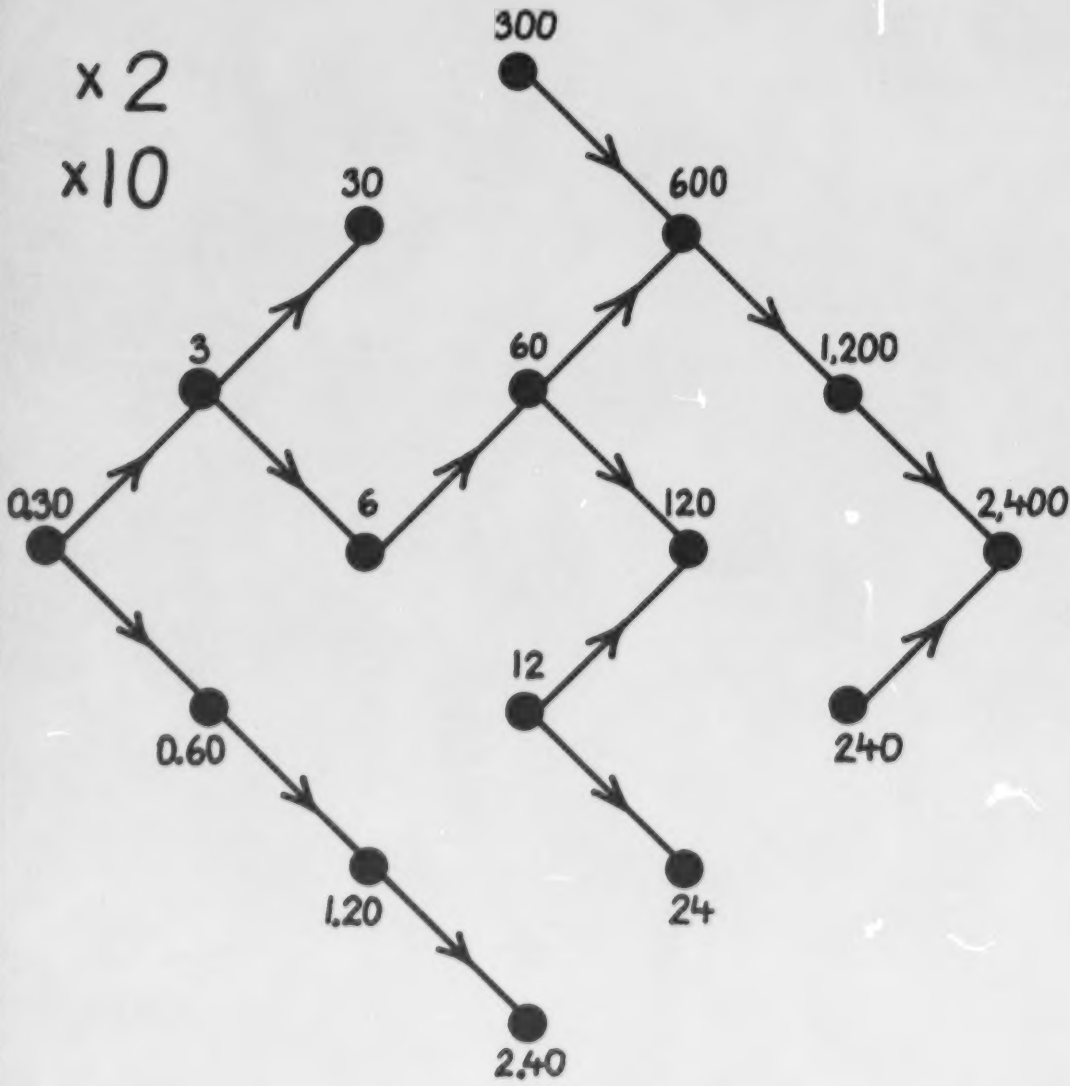
● Exercise 2:

Draw on the board:

● $\times 2$
 $\times 10$

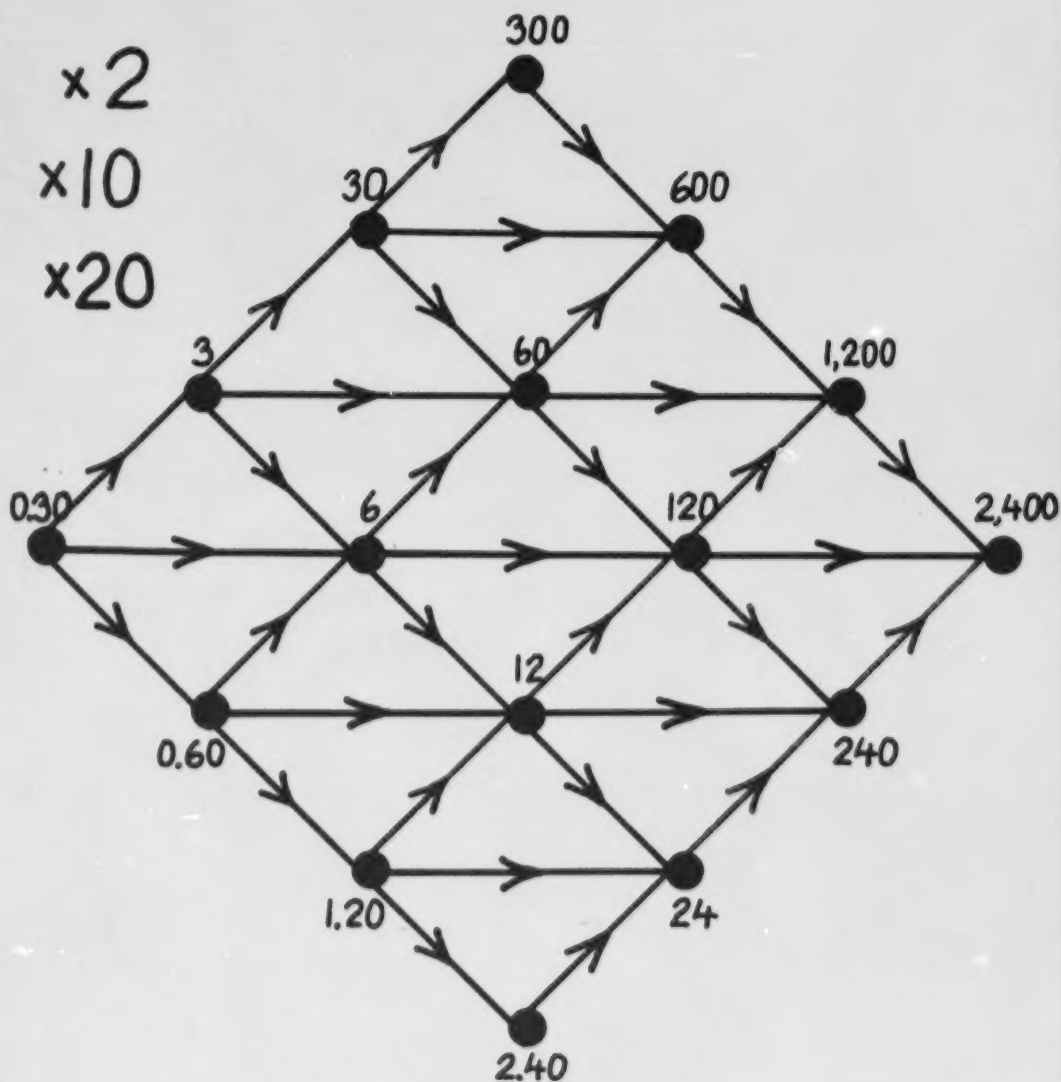


Let the students take turns labeling the dots on this picture at the board.
Accept any order of labeling.



Afterwards, ask them to draw successively all the missing arrows, red and blue, and to use green to add as many $\times 20$ arrows as they can.

Solution:



Write on the board:

$$20 \times 0.30 =$$

$$20 \times 120 =$$

$$12 \div 20 =$$

$$6 \div 20 =$$

T: The arrow picture can help you to complete these calculations. Do it on your sheet of paper.

After a while, ask some students to explain their answers.

$$20 \times 0.30 = 6$$

$$20 \times 120 = 2,400$$

$$12 \div 20 = 0.6$$

$$6 \div 20 = 0.30$$

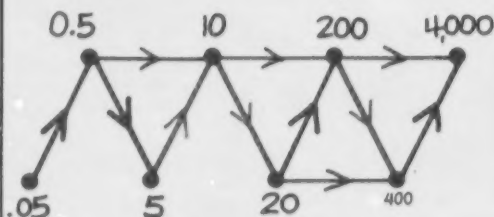
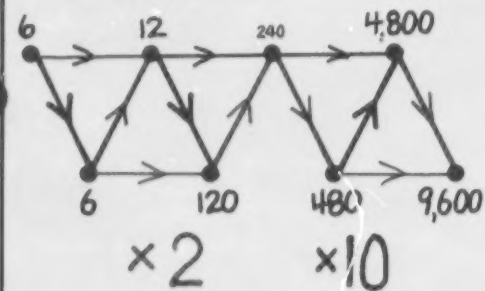
Worksheets N34, N35 and N36 are available for the rest of the lesson.

Name _____

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N 34 *

Label all the dots. Use green to draw as many $\times 20$ arrows as you can.

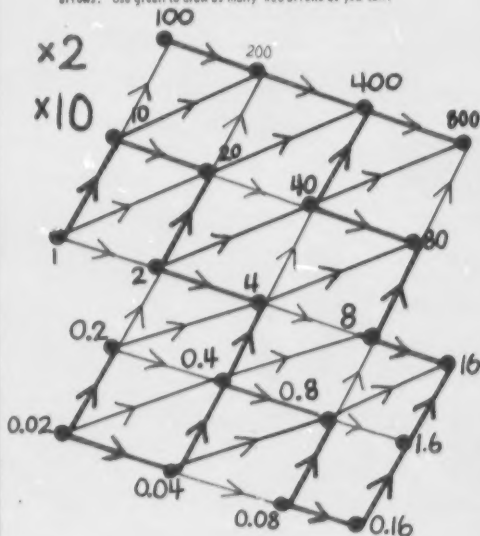


Name _____

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N 35 **

Label all the dots. Draw the missing red arrows and the missing blue arrows. Use green to draw as many $\times 20$ arrows as you can.

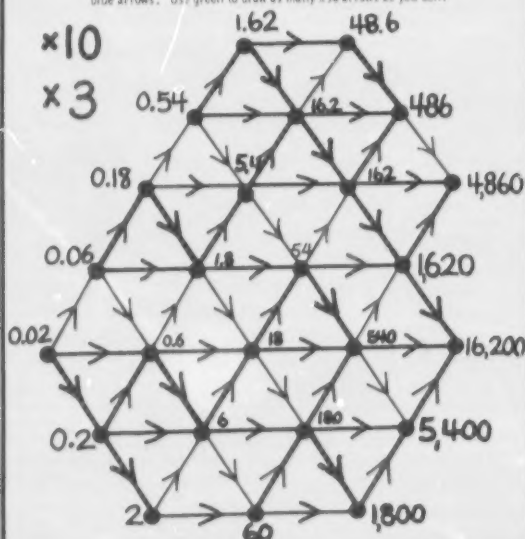


Name _____

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N 36 ***

Label all the dots. Draw the missing red arrows and the missing blue arrows. Use green to draw as many $\times 30$ arrows as you can.



3

GUESS MY RULE

A short reminder of the game from the storybook DANCING FRIENDS^{*} serves to introduce the extraordinarily rich game Guess My Rule. The basic idea is that the teacher, and then the students, invent new operations which are usually, but not necessarily, based on the standard arithmetic ones. The class cooperates in attempting to determine the "secret" operation. For example, students have invented operations "*" such as:

I. $2 * 4 = 5$

$4 * 2 = 4$

$0 * 2 = 2$

$2 * 0 = 1$

Rule: Add the second number to half of the first number.

II. $2 * 4 = 4$

$4 * 2 = 6$

$0 * 2 = -2$

$2 * 0 = 0$

Rule: Subtract the second number from the product of the two numbers.

In the context of working with these "new" operations, a great deal of mental arithmetic and equation solving is accomplished in a stimulating setting.

CAPSULE LESSON SUMMARIES

Section 3.1: Review the operation from DANCING FRIENDS, and then introduce the Guess My Rule game using the operation * where * is defined as: double the first number and then add the second number. As a supplementary activity, suggest that students invent their own operations. (Worksheets N37, N38)

Section 3.2: Play Guess My Rule with the operation * where * is defined as: double the first number and then add one-half of the second number. (Worksheets N39, N40, N41)

^{*} DANCING FRIENDS is one of the storybooks used in CSMP Mathematics for the Upper Primary Grades, Part II (3rd grade).

3.1 Guess My Rule #1

T: Do you remember the calculation that Zero introduced in the DANCING FRIENDS storybook?

Write on the board:

$$7 \oplus 5 =$$

Have a student explain why the answer is "2".

T: Today, I shall introduce a new calculation and you will have to guess what my rule is.

Write on the board:

$$5 * 3 = 13$$

$$3 * 5 = 11$$

$$1 * 5 =$$

T: Who can guess which number $1 * 5$ is?

Perhaps some students will guess that it's 7. In any case, write "7" on the board.

T: If you have already guessed my rule, keep it secret for the moment.

Write on the board:

$$5 * 1 =$$

T: Who can guess which number $5 * 1$ is?

Probably some students will guess that it's 11. In any case, write "11" on the board.

Follow the same steps for the next calculations:

$$3 * 0 = 6$$

$$6 * 3 = 15$$

$$0 * 3 = 3$$

$$3 * 6 = 12$$

$$5 * 2 = 12$$

$$10 * 4 = 24$$

$$2 * 5 = 9$$

$$4 * 10 = 18$$

When at least half of your students have guessed your rule, have one of them explain that in each of these problems you double the first number and then add the second number.

Write on the board:

★		
7	*	1 =
1	*	7 =
20	*	5 =
5	*	20 =
50	*	25 =
25	*	50 =
-2	*	3 =
3	*	-2 =
-5	*	2 =
2	*	-5 =

★★		
4	*	= 10
	*	4 = 10
8	*	= 20
	*	8 = 20
5	*	= 55
	*	5 = 55
3	*	= 103
	*	3 = 103

Have the students solve as many problems as they can by themselves and have them progressively write the answers on the board:

★				★★			
7	*	1	= 15	4	*	2	= 10
1	*	7	= 9	3	*	4	= 10
20	*	5	= 45	8	*	4	= 20
5	*	20	= 30	6	*	8	= 20
50	*	25	= 125	5	*	45	= 55
25	*	50	= 100	25	*	5	= 55
-2	*	3	= -1	3	*	97	= 103
3	*	-2	= 4	50	*	3	= 103
-5	*	2	= -8				
2	*	-5	= -1				

Write on the board:

$$\square * \square = 18$$

T: The same number is in each of these frames. What is this number?

Have students whisper the answer in your ear. Have one of them explain to the class why the number is six.

$$\square 6 * \square 6 = 18$$

Follow the same steps for:

$$\square * \square = 30$$

$$\square * \square = 39$$

$$\square * \square = 24$$

$$\square * \square = 75$$

Your students will surely notice that in each problem, the hidden number is one-third of the number on the right of the equal sign.

$$10 * 10 = 30$$

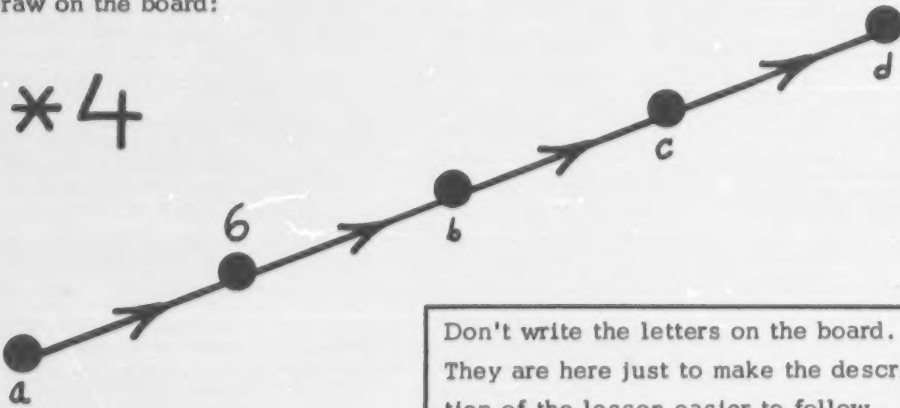
$$13 * 13 = 39$$

$$8 * 8 = 24$$

$$25 * 25 = 75$$

Draw on the board:

$* 4$



Don't write the letters on the board.
They are here just to make the description of the lesson easier to follow.

4F

Point to b and ask:

T: How can we find this number?

By following the arrow starting from 6, have a student explain that this number has to be $6 * 4 = 16$.

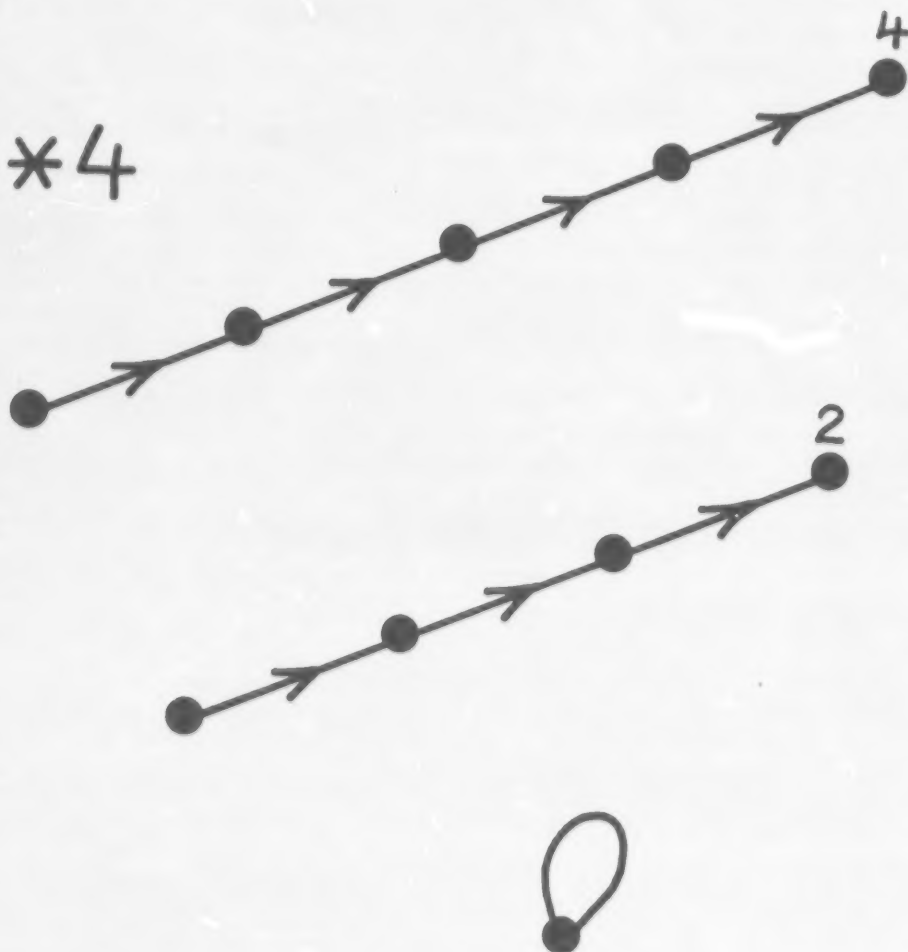
Write "16" on the picture and have the students calculate the three other numbers.

Solution:



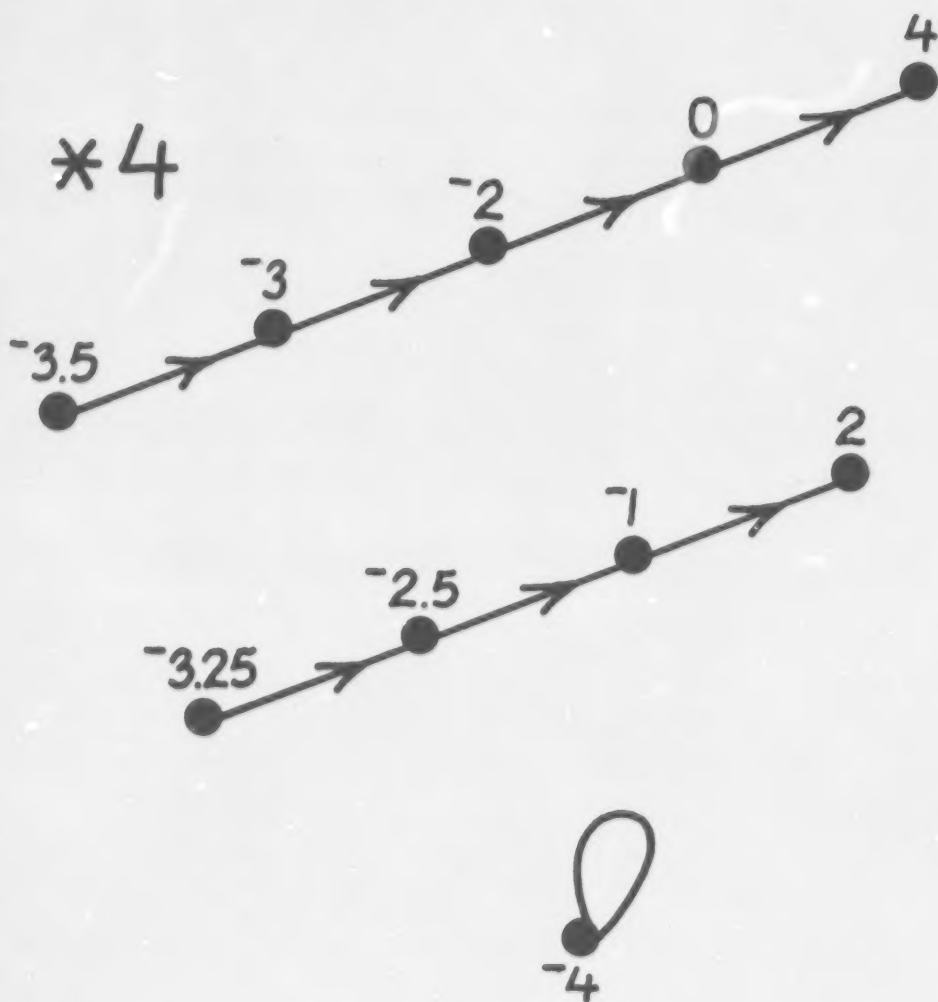
Worksheets N37 and N38 are available.

If you have some very fast workers who need a *** problem, draw this picture on the board, ask them to copy it on a sheet of paper and to label the dots:



4F

Solution to the *** problem:



Name _____

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N37 *

Complete:

$$6 * 2 = 12$$

$$2 * 6 = 12$$

$$10 * 6 = 26$$

$$6 * 10 = 22$$

$$50 * 4 = 104$$

$$4 * 50 = 58$$

$$100 * 25 = 225$$

$$25 * 100 = 150$$

$$7 * 6 = 28$$

$$6 * 8 = 20$$

$$6 * 2 = 10$$

$$10 * 7 = 27$$

$$7 * 13 = 27$$

$$0 * 6 = 6$$

$$6 * 6 = 6$$

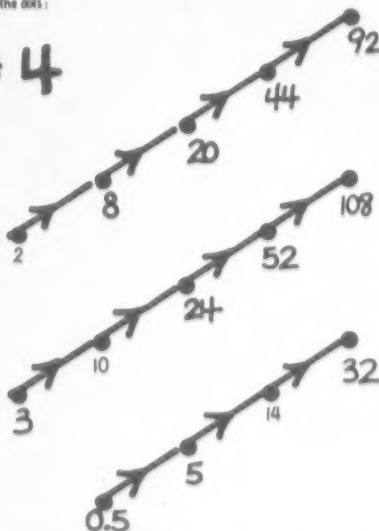
Name _____

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N38 **

Label the dots:

* 4



APPENDIX: Short Supplementary Activity

After the first lesson on "Guess My Rule", you could devote 10 to 15 minutes from time to time to the following activity which seems to be enjoyable and productive for the students.

T: Do you remember the * operation that we introduced some days ago?

Write on the board:

$$5 * 7 =$$

$$7 * 5 =$$

Have a student explain that we had to double the first number and then add the second number. So,

$$5 * 7 = 17$$

$$7 * 5 = 19$$

T: Today you will try to invent your own calculation and the other students will have to guess what your rule is.

Have the students think for a little while.

T: Who has invented a new calculation?

From the volunteers, choose a student who has a high numerical ability. It's obviously impossible to predict what will happen in your class. As an example, we will describe what happened in two of our fourth grade classes.

In one group, Angela was chosen and the teacher wrote on the board:

$$4 * 2 =$$

$$2 * 4 =$$

T: For Angela's calculation, we will use the same symbol * as before but with a new meaning. It's just like the way we use the arrows. One day, an arrow can be for $\times 2$ and on another day for $+10$.

The teacher asked Angela to complete the calculations:

$$4 * 2 = 4$$

$$2 * 4 = 5$$

The teacher wrote on the board:

$$0 * 2 =$$

$$2 * 0 =$$

and some students guessed the answers:

$$0 * 2 = 2$$

$$2 * 0 = 1$$

The teacher followed the same steps for these problems:

$$10 * 2 = 7$$

$$3 * 3 = 4\frac{1}{2} = 4.5$$

$$2 * 10 = 11$$

$$1 * 4 = 4\frac{1}{2} = 4.5$$

$$6 * 4 = 7$$

$$4 * 1 = 3$$

$$4 * 6 = 8$$

At this point, most of the students had guessed the rule and one of them explained that you had to calculate half of the first number and then add the second number.

The teacher wrote these problems on the board and asked the students to solve as many of them as they could by themselves:

★	★★
$10 * 4 =$	$\square * 6 = 10$
$4 * 10 =$	$6 * \square = 10$
$20 * 5 =$	$\square * 12 = 20$
$5 * 20 =$	$12 * \square = 20$
$6 * 6 =$	$\square * 3 = 5\frac{1}{2}$
$7 * 100 =$	$3 * \square = 5\frac{1}{2}$
$100 * 7 =$	

Afterwards, different students were asked to write their answers on the board and explain them, if necessary.

★	★★
$10 * 4 = 9$	$\square 8 * 6 = 10$
$4 * 10 = 12$	$6 * \square 7 = 10$
$20 * 5 = 15$	$\square 16 * 12 = 20$
$5 * 20 = 22.5 = 22\frac{1}{2}$	$12 * \square 14 = 20$
$6 * 6 = 9$	$\square 5 * 3 = 5\frac{1}{2} = 5.5$
$7 * 100 = 103.5 = 103\frac{1}{2}$	$3 * \square 4 = 5\frac{1}{2} = 5.5$
$100 * 7 = 57$	

In the other group, Vonda was chosen and it took more time to guess her rule.

First she gave:

$$5 * 2 = 8$$

$$2 * 5 = 5$$

Then:

$$4 * 6 = 18$$

$$6 * 4 = 20$$

$$5 * 0 = 0$$

$$0 * 5 = -5$$

$$10 * 1 = 9$$

$$1 * 10 = 0$$

At this point, one student explained that Vonda was multiplying the two numbers and then subtracting the second number. The rule was applied to check the examples above:

$$5 * 2 = (5 \times 2) - 2 = 8$$

$$4 * 6 = (4 \times 6) - 6 = 18$$

$$5 * 0 = (5 \times 0) - 0 = 0$$

$$10 * 1 = (10 \times 1) - 1 = 9$$

$$2 * 5 = (2 \times 5) - 5 = 5$$

$$6 * 4 = (6 \times 4) - 4 = 20$$

$$0 * 5 = (0 \times 5) - 5 = -5$$

$$1 * 10 = (1 \times 10) - 10 = 0$$

The teacher wrote these problems on the board and asked the students to solve as many of them as they could by themselves:

$$3 * 4 =$$

$$4 * 3 =$$

$$5 * 5 =$$

$$5 * 6 =$$

$$6 * 5 =$$

$$6 * 6 =$$

$$4 * 7 =$$

$$7 * 4 =$$

$$5 * 8 =$$

$$8 * 5 =$$

$$7 * 7 =$$

$$9 * 9 =$$

$$4 * \square = 15$$

$$\square * 4 = 16$$

$$4 * \square = 24$$

$$\square * 4 = 24$$

$$5 * \square = 8$$

$$\square * 5 = 25$$

$$7 * \square = 36$$

$$\square * 7 = 35$$

Some students found an easier way of solving Vonda's problems: subtract 1 from the first number and then multiply the result by the second number. (These two rules are, in fact, equivalent.)

Answers:

$$3 * 4 = 8$$

$$4 * 7 = 21$$

$$4 * 3 = 9$$

$$7 * 4 = 24$$

$$5 * 5 = 20$$

$$5 * 8 = 32$$

$$5 * 6 = 24$$

$$8 * 5 = 35$$

$$6 * 5 = 25$$

$$7 * 7 = 42$$

$$6 * 6 = 30$$

$$9 * 9 = 72$$

$$4 * \boxed{5} = 15$$

$$5 * \boxed{2} = 8$$

$$\boxed{5} * 4 = 16$$

$$\boxed{6} * 5 = 25$$

$$4 * \boxed{8} = 24$$

$$7 * \boxed{6} = 36$$

$$\boxed{7} * 4 = 24$$

$$\boxed{6} * 7 = 35$$

3.2 Guess My Rule #2Exercise 1:

T: Today, I shall introduce a new calculation and you will have to guess my rule.

Write on the board:

$$\begin{array}{rcl} 10 * 4 & = & 22 \\ 4 * 10 & = & 13 \\ 100 * 4 & = & 202 \\ 4 * 100 & = & \end{array}$$

T: Who can guess the number $4 * 100$?

If nobody guesses that $4 * 100 = 58$, give the answer yourself.

Write on the board:

$$\begin{array}{rcl} 0 * 4 & = & \\ 4 * 0 & = & \end{array}$$

T: Who can guess these numbers? If you have already guessed my rule, keep it a secret for the moment.

With or without the help of your students, complete these calculations:

$$0 * 4 = 2$$

$$4 * 0 = 8$$

Follow the same steps for the next calculations:

$$20 * 0 = 40$$

$$20 * 2 = 41$$

$$0 * 20 = 10$$

$$2 * 20 = 14$$

At this point, let several students give a rule they think will work and then check all the calculations on the board to see if the rule is correct. If no one has guessed the rule correctly, give it (double the first number and then add half the second number) yourself and again check all the calculations already done to verify it.

Write on the board:

★	★★
$10 * 10 =$	$8 * 7 =$
$20 * 2 =$	$7 * 8 =$
$2 * 20 =$	$5 * 5 =$
$40 * 8 =$	$15 * 15 =$
$8 * 40 =$	$-2 * 6 =$
$50 * 50 =$	$6 * -2 =$

Have the students solve as many problems as they can by themselves and have them progressively write the answers on the board:

★	★★
$10 * 10 = 25$	$8 * 7 = 19.5$
$20 * 2 = 41$	$7 * 8 = 18$
$2 * 20 = 14$	$5 * 5 = 12.5$
$40 * 8 = 84$	$15 * 15 = 37.5$
$8 * 40 = 36$	$-2 * 6 = -1$
$50 * 50 = 125$	$6 * -2 = 11$

Exercise 2:

Write on the board:

$$\square * 6 = 13$$

$$6 * \triangle = 13$$

and have the students complete the calculations on their sheets of paper:

$$\boxed{5} * 6 = 13$$

$$6 * \triangle 2 = 13$$

Write on the board:

★	★★
$\square * 10 = 25$	$\square * 4 = -10$
$10 * \triangle = 25$	$4 * \triangle = -10$

Have the students try to solve these problems by themselves. After a while, ask some of them to explain the answers to the whole group.

★

$$\begin{array}{l} \boxed{10} * 10 = 25 \\ 10 * \triangle 10 = 25 \end{array}$$

★★

$$\begin{array}{l} \boxed{-6} * 4 = -10 \\ 4 * \triangle -36 = -10 \end{array}$$

Exercise 3:

Write on the board:

$$\square * \triangle = 10$$

$\square$	$\triangle$

T: What numbers could be in each of these frames? Many solutions are possible. Try to find at least one of them.

Write in the chart some solutions that are suggested by your students. For instance,

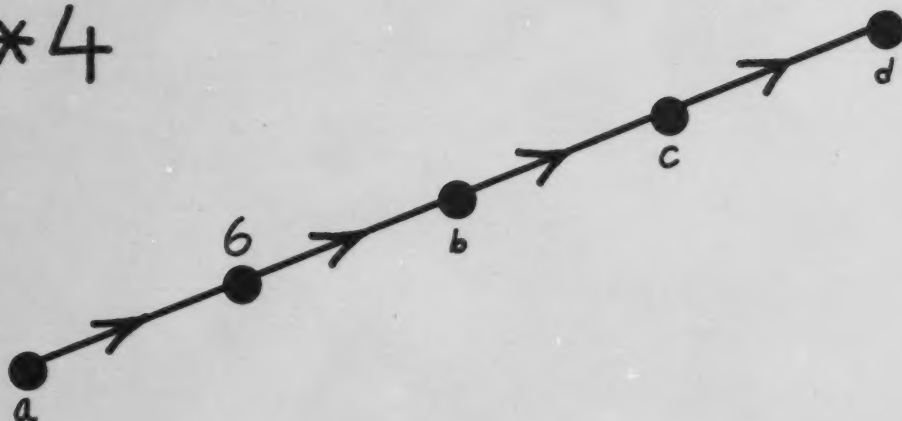
	
5	0
0	20
4	4
10	-20

Ask them to copy the chart on a sheet of paper and to try to find some more solutions.

After a while, have students explain some of their answers to the whole group.

Exercise 4:

Draw on the board:

 6×4 

Don't write the letters on the board.
They are here just to make the description
of the lesson easier to follow.

Point to b and ask:

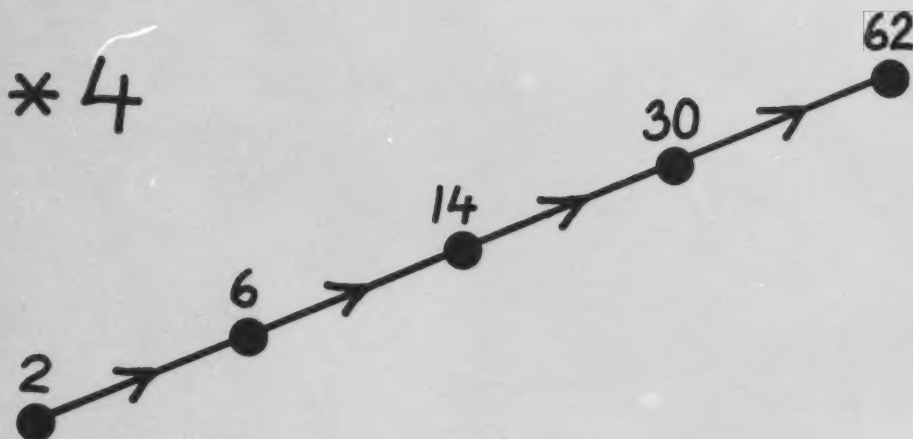
T: How can we find this number?

By following the arrow starting from 6, have a student explain that this number has to be $6 \times 4 = 14$.

14F

Write "14" on the picture and have the students calculate the three other numbers.

Solution:



Worksheets N39, N40 and N41 are available for the rest of the period.

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Name _____

N 39 *

Complete:

10 * 40 = 40	2 * 20 = 14
40 * 10 = 85	30 * 10 = 65
100 * 20 = 210	5 * 40 = 30
20 * 100 = 90	50 * 4 = 102
8 * 8 = 20	50 * 10 = 105
12 * 12 = 30	7 * 12 = 20
30 * 30 = 75	

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Name _____

N 40 **

Label the dots:

*8

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Name _____

N 41 ***

$\square * \triangle = 20$

Complete the charts:

$\square$	$\triangle$
6	16
8	8
5	20
2	32
75	100
10	80
$75 = 7\frac{1}{2}$	10
10	0
$125 = 12\frac{1}{2}$	70

$\square$	$\triangle$
9	4
4	24
7	12
12	8
5	60
20	40
0	40
1	36
$85 = 8\frac{1}{2}$	6

N41

(Many other solutions are possible.)

TEN TIMES

A gentle introduction, via the Minicomputer and mental arithmetic, reminds the students of the effect of multiplying a number by 10. Then, using both the Minicomputer and arrow diagrams, progressively richer experiences are presented involving addition functions and multiplying by 10, including work with decimals.

This setting is used to pose such problems as:

1. Using exactly two $\times 10$ and ten $+1$ arrows, starting from 0, find the largest (smallest) numbers possible for the end of the "road." Find all possible ending numbers between the largest and smallest possible ending numbers.
2. Using $+1$ and $\times 10$ arrows build the shortest possible road from 0 to 356.

CAPSULE LESSON SUMMARIES

Section 4.1: With the support of the Minicomputer, review the effect of multiplying numbers by 10. Follow this with a mental arithmetic activity. Explore situations which use addition and $\times 10$ functions alternately. Let students build roads from 0 to 356 with $\times 10$ and $+1$ arrows and try to find the minimum number of arrows necessary.

Section 4.2: Continue multiplying decimal numbers by 10, using the Minicomputer as support. Investigate which numbers can be at the end of roads starting from 0 and using exactly two $\times 10$ and ten $+1$ arrows.

4.1 Ten Times #1Exercise 1:

Put on the board:

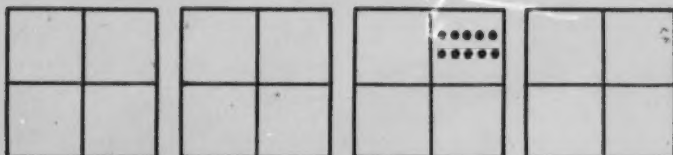


T: What is this number?

S: $10 \times 4 = 40$.

Have students perform robot moves^{*} on the Minicomputer and finally get one checker on the 40-square.

Put on the board:



T: What is this number?

S: $10 \times 40 = 400$.

T: Is it interesting to make robot moves again?

^{*} See STANDARD MOVES ON THE MINICOMPUTER in the introduction to the World of Numbers strand.

S: No. It will be exactly like before and we will get one checker on the 400-square.

Write on the board:

$$10 \times 400 = \quad 10 \times 4,000 =$$

and have the students complete these calculations.

$$10 \times 400 = 4,000 \quad 10 \times 4,000 = 40,000$$

Exercise 2: (Mental Arithmetic)

$$10 \times 2; \quad 10 \times 20; \quad 10 \times 200; \quad 10 \times 2,000$$

$$10 \times 8; \quad 10 \times 80; \quad 10 \times 800; \quad 10 \times 8,000$$

$$10 \times 3; \quad 10 \times 30; \quad 10 \times 300; \quad 10 \times 3,000$$

$$10 \times 7; \quad 10 \times 70; \quad 10 \times 700; \quad 10 \times 7,000$$

Exercise 3:

Write on the board:

$$10 \times 5 =$$

$$10 \times 30 =$$

$$10 \times 35 =$$

and have the students solve these problems on a sheet of paper. Then ask one student to complete the calculations on the board:

$$10 \times 5 = 50$$

$$10 \times 30 = 300$$

$$10 \times 35 = 350$$

Follow the same steps for:

$$10 \times 60 =$$

$$10 \times 400 =$$

$$10 \times 460 =$$

$$10 \times 20 =$$

$$10 \times 6,000 =$$

$$10 \times 6,020 =$$

$$10 \times 7 =$$

$$10 \times 200 =$$

$$10 \times 207 =$$

$$10 \times 6 =$$

$$10 \times 600 =$$

$$10 \times 5,000 =$$

$$10 \times 5,606 =$$

Answers:

$$10 \times 60 = 600$$

$$10 \times 400 = 4,000$$

$$10 \times 460 = 4,600$$

$$10 \times 20 = 200$$

$$10 \times 6,000 = 60,000$$

$$10 \times 6,020 = 60,200$$

$$10 \times 7 = 70$$

$$10 \times 200 = 2,000$$

$$10 \times 207 = 2,070$$

$$10 \times 6 = 60$$

$$10 \times 600 = 6,000$$

$$10 \times 5,000 = 50,000$$

$$10 \times 5,606 = 56,060$$

Exercise 4:

Display one Minicomputer board:



T: What is this number?

S: 0.

Draw on the board:

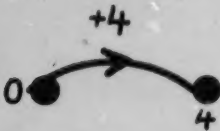
0 ●

Put one checker on the 4-square.

T: What number did I put on the Minicomputer?

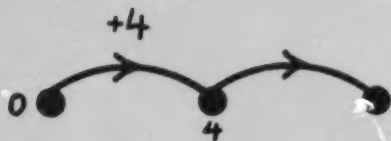
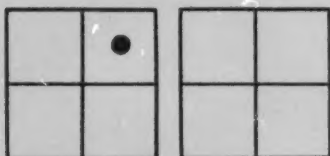
S: Four.

Draw on the board:



5F

Add one board to the right of the first board and draw this red arrow:



T: What number is on the Minicomputer now?

S: 40.

T: What could this red arrow be for?

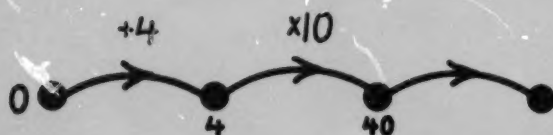
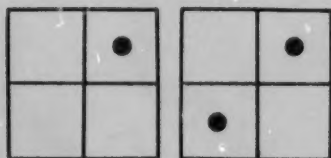
S: $\times 10$.

S: +36.

T: Both are correct. I'm going to let it be for $\times 10$.

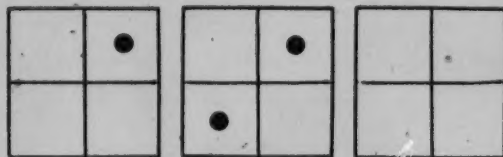
Write " $\times 10$ " above the red arrow.

Add these checkers on the boards and draw this blue arrow:

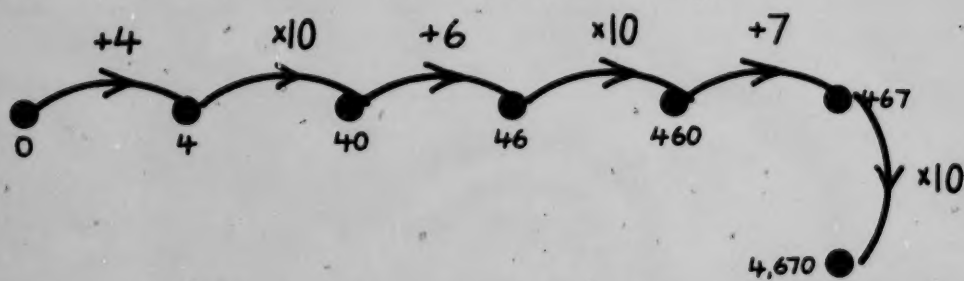
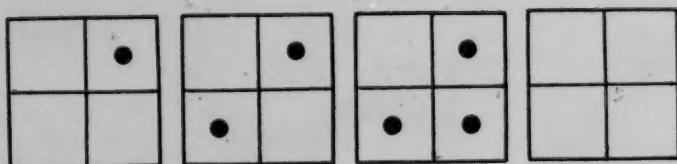
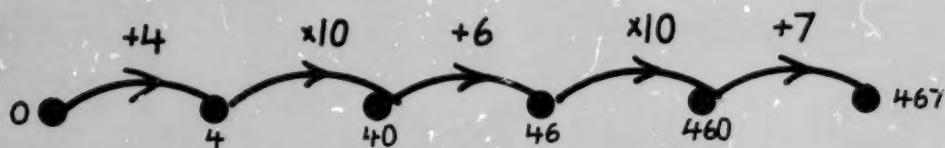
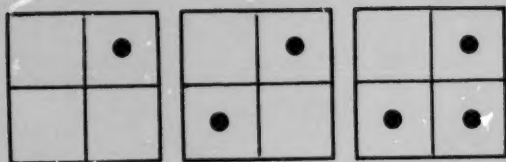


The students should notice that the number on the Minicomputer is 46 and that the blue arrow could be for $+6$. Label the arrow " $+6$ " in blue and label the dot "46" in the arrow picture.

Follow similar steps successively for:

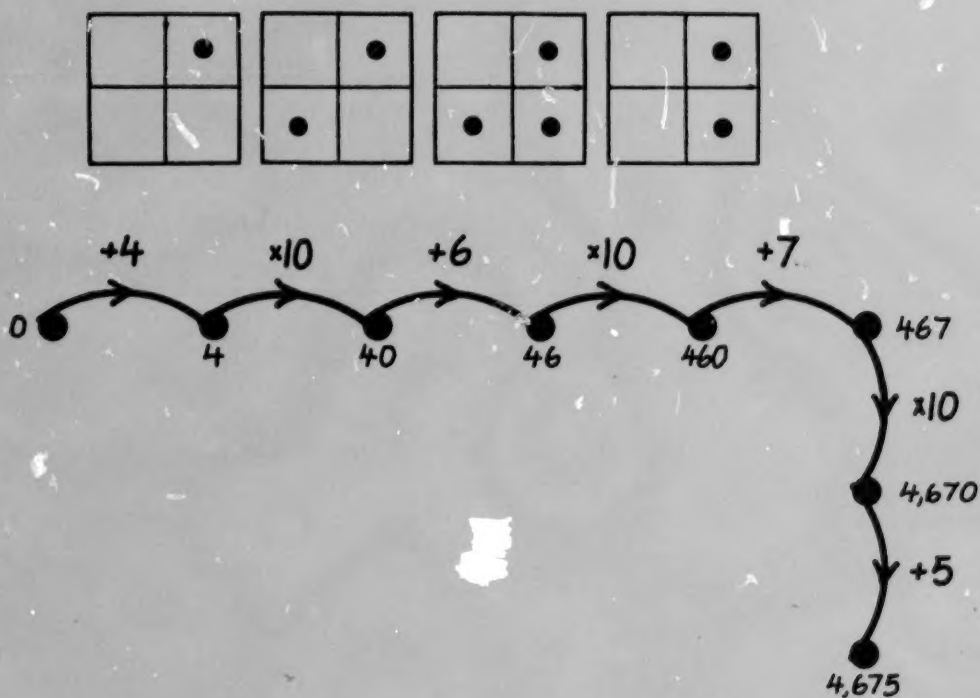


5F



100

3.7



T: And now we are going to come back to where we started.

Remove the checkers from the ones' board and ask:

T: What number is on the Minicomputer now?

S: 4,670.

T: How can we show what I have done in the arrow picture?

S: Draw a return -5 arrow.

Have a student draw a -5 return arrow in yellow from 4,675 to 4,670.

Then successively: take away the last board on the right and have a student draw a +10 return arrow in green from 4,670 to 467.

Remove the three checkers from the board on the right and have a student draw a -7 return arrow in yellow from 467 to 460.

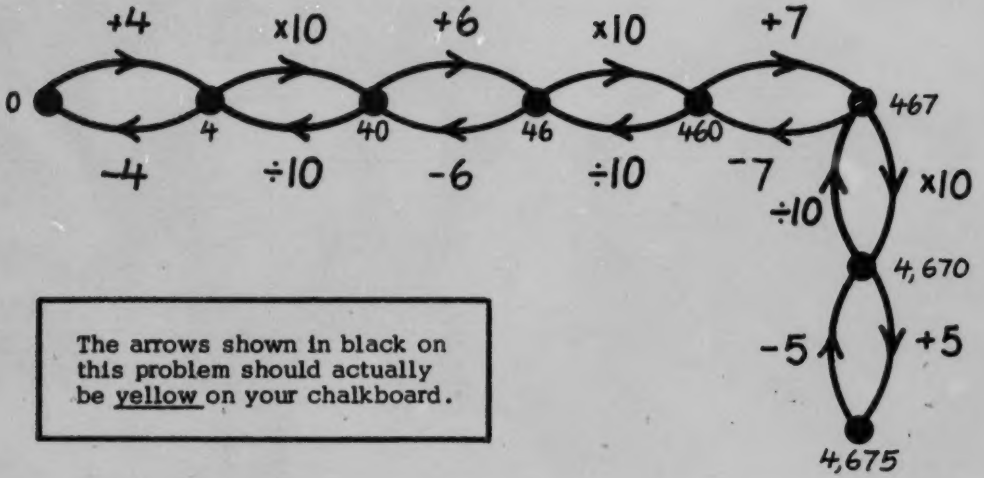
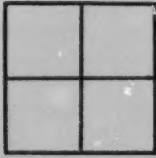
Take away the last board on the right and have a student draw a +10 return arrow in green from 460 to 46.

Remove the two checkers from the board on the right and have a student draw a -6 return arrow in yellow from 46 to 40.

Take away the last board on the right and have a student draw a +10 return arrow in green from 40 to 4.

Remove the checker from the last board and have a student draw a -4 return arrow in yellow from 4 to 0.

At this point, your board should look like this:



The arrows shown in black on this problem should actually be yellow on your chalkboard.

Exercise 5:

Draw on the board:



356

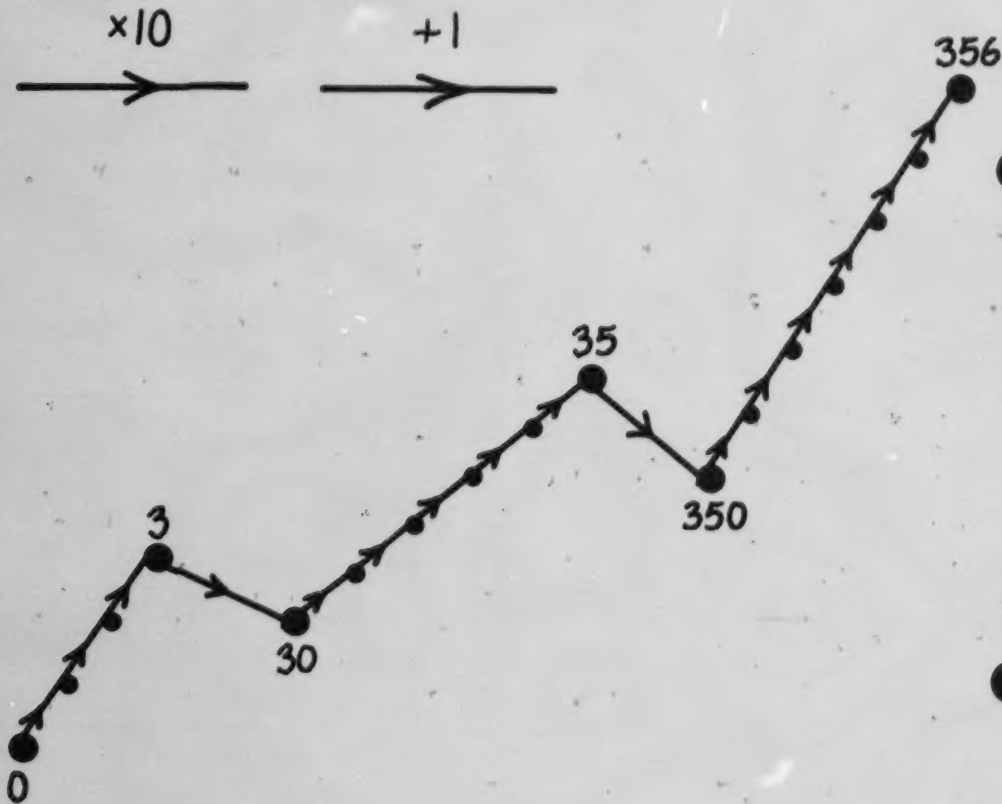
T: Build a blue-red road from 0 to 356. Try to use as few arrows as possible.

Let the students work on their own for a while. Walk around and observe their work. As soon as the children have solved the problem, ask them to build a blue-red road from 0 to 6,034 (by using as few arrows as possible).

Before the end of the lesson, ask the students how many arrows they used in solving the first problem. Probably many of them will answer "16" and some, a number larger than 16.

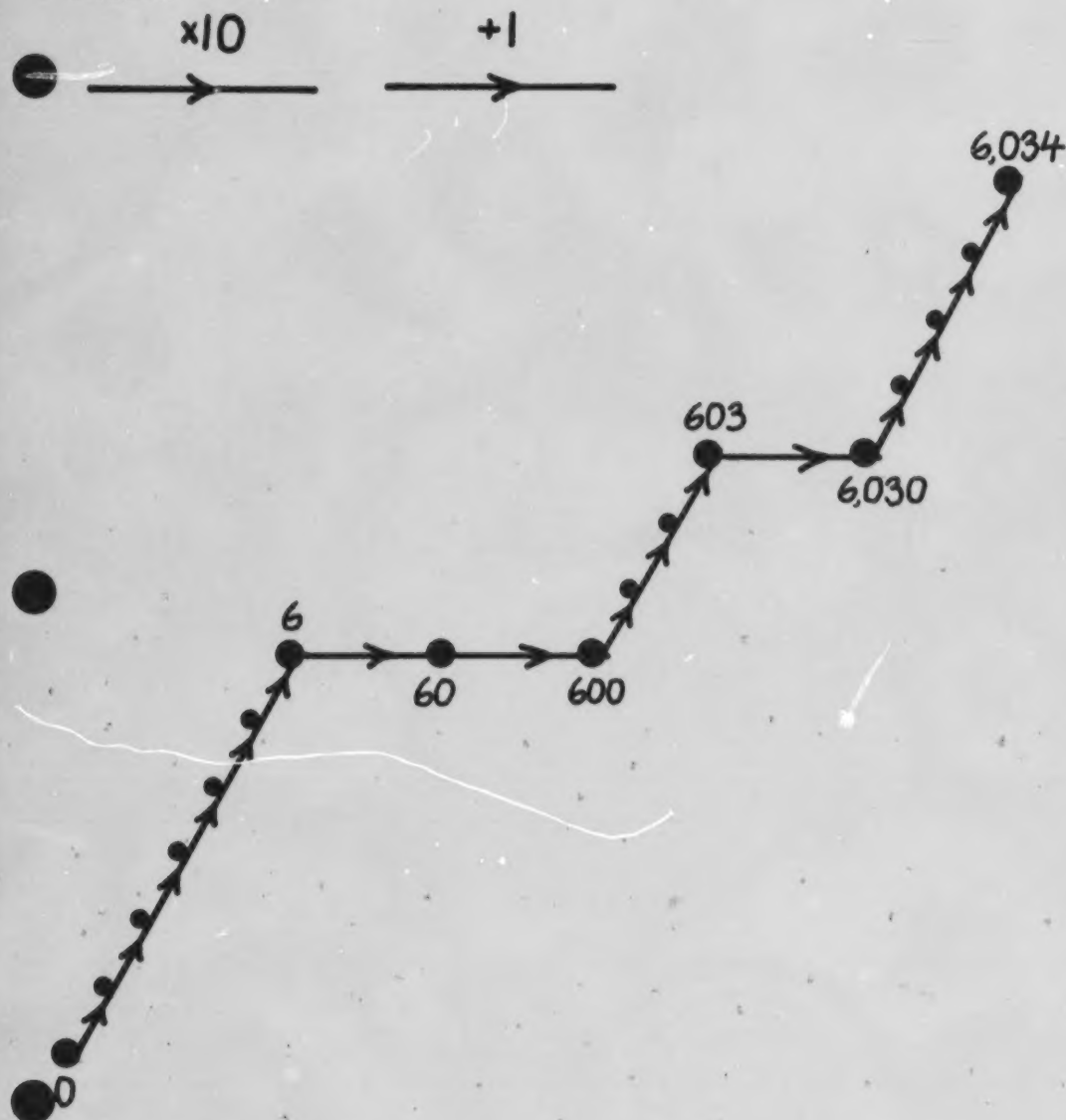
Ask a student to draw a road with 16 arrows on the board. In his or her picture, enlarge the dots for 0, 3, 30, 35, 350 and 356.

Solution:



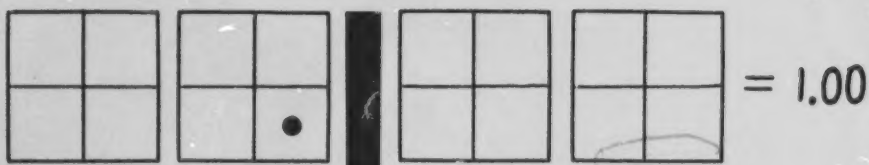
If you still have time, have another student draw the solution to the second problem. (The minimum number of arrows for this problem is also 16.)

Solution:



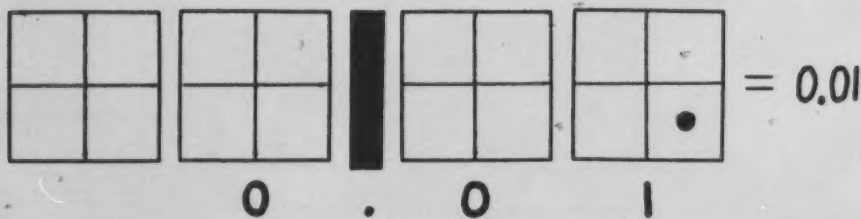
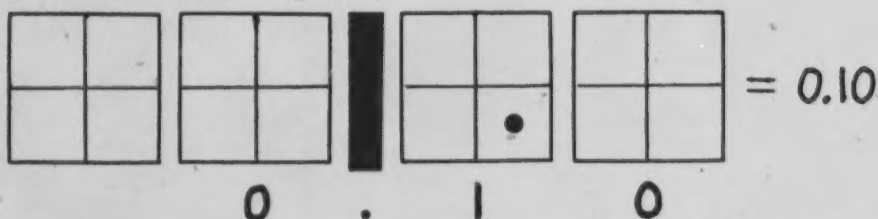
4.2 Ten Times #2Exercise 1:

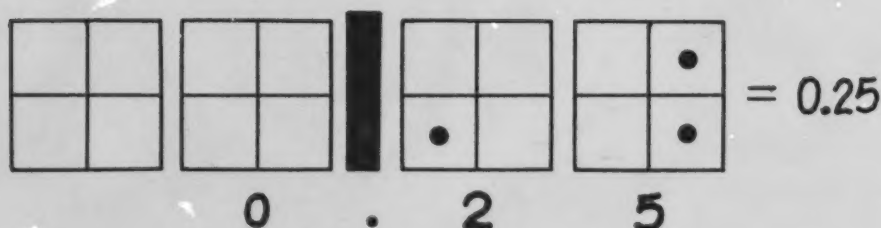
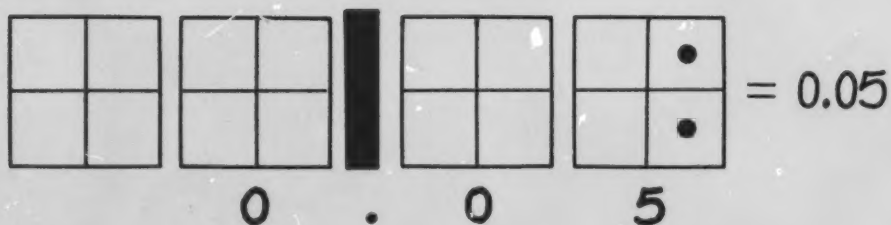
Put on the board:



T: I have put \$1 on the Minicomputer.

Have students put on successively one dime, one penny, one nickel and one quarter:





Put these ten checkers on the board:



T: How much money is there on the Minicomputer?

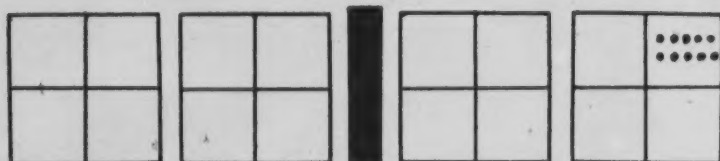
Have students whisper the answer (\$4) in your ear. Then have students make the forward and backward robot moves to actually show that the answer is 4. To reinforce the understanding of these moves, have the students verbalize them in this way:

$$\begin{aligned}
 4 \text{ dimes} &= 2 \text{ dimes} + 2 \text{ dimes} \\
 4 \text{ dimes} + 4 \text{ dimes} &= 8 \text{ dimes} \\
 8 \text{ dimes} + 2 \text{ dimes} &= 1 \text{ dollar}
 \end{aligned}$$

Write on the board:

$$10 \times 0.40 = 4.00$$

Put these ten checkers on the board:

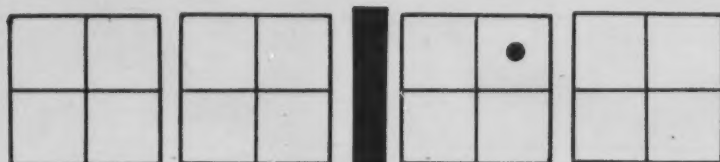


T: How much money is there on the Minicomputer?

Have students whisper the answer (40¢ or 4 dimes) in your ear.

T: How can we write the answer with one checker?

Have a student put this checker on the "dimes" board.



Emphasize the fact that

ten checkers on a purple square

=

one checker on the purple square
of the next board to the left

or more generally,

ten checkers on the same square
of one board

=

one checker on the square of the
same color of the next board to
the left

Write on the board:

$$10 \times 0.04 = 0.40 = 0.4 = .4$$

Ask the students to copy the following calculations on a sheet of paper and to solve them:

$$10 \times 0.05 =$$

$$10 \times 0.20 =$$

$$10 \times 0.25 =$$

Have students explain the answers ($0.50 = 0.5$; $2.00 = 2.0 = 2$; $2.50 = 2.5$) using the mental support of money if it's needed.

Follow similar steps for these problems:

$$10 \times 0.04 =$$

(Answer: $0.4 = 0.40$)

$$10 \times 0.30 =$$

(Answer: $3.00 = 3.0 = 3$)

$$10 \times 27.00 =$$

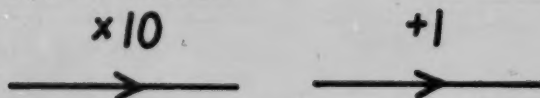
(Answer: $270.00 = 270.0 = 270$)

$$10 \times 27.34 =$$

(Answer: $273.4 = 273.40$)

Exercise 2:

Draw on the board:



Use exactly:

2 red arrows

10 blue arrows

0

T: Starting from 0 and using exactly 10 blue arrows and 2 red arrows, try to get the largest ending number possible.

Let the students work on their own. As soon as a student has solved the problem (answer: 1,000), tell him or her:

T: Starting from 0 and using exactly 10 blue arrows and two red arrows, try to get the smallest ending number possible. In this problem, don't use any red loops.

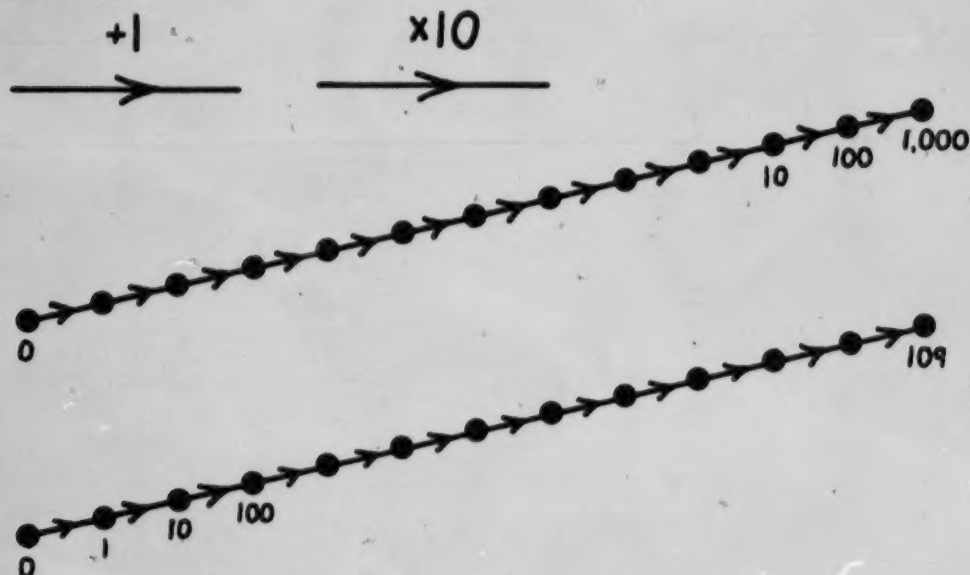
As soon as a student has solved the problem (answer: 109) tell him or her:

T: Starting from 0 and using exactly 10 blue arrows and two red arrows, try to get an ending number between the largest ending number possible and the smallest ending number possible.

If you have time, draw two or three different solutions.

After a while have students draw on the board the solutions for the first two problems.

Solutions:



Then ask students who have solved the third problem to give you the ending numbers of their roads. Write them on the board. Some possible answers are:

802 ; 424 ; 307 ; 712 ; 325 ;
640 ; 460 ; 730 ; 271 ; 901

T: Look at these numbers. What do you notice?

Let the students express their ideas. Probably they will notice that all the numbers have three digits and that the sum of these three digits is always 10.

T: Can you give me some more numbers like that?

Write on the board the students' answers that could be ending numbers, for instance,

136 ; 208 ; 244 ; 523 ; and so on.

Choose one of these numbers, for instance 523 and ask:

T: Starting from 0 and using exactly two red arrows and ten blue arrows, try to get the number 523.

After a while, ask :

T: How did you solve the problem? Did you use blue or red arrows first?

S: I used five blue arrows first.

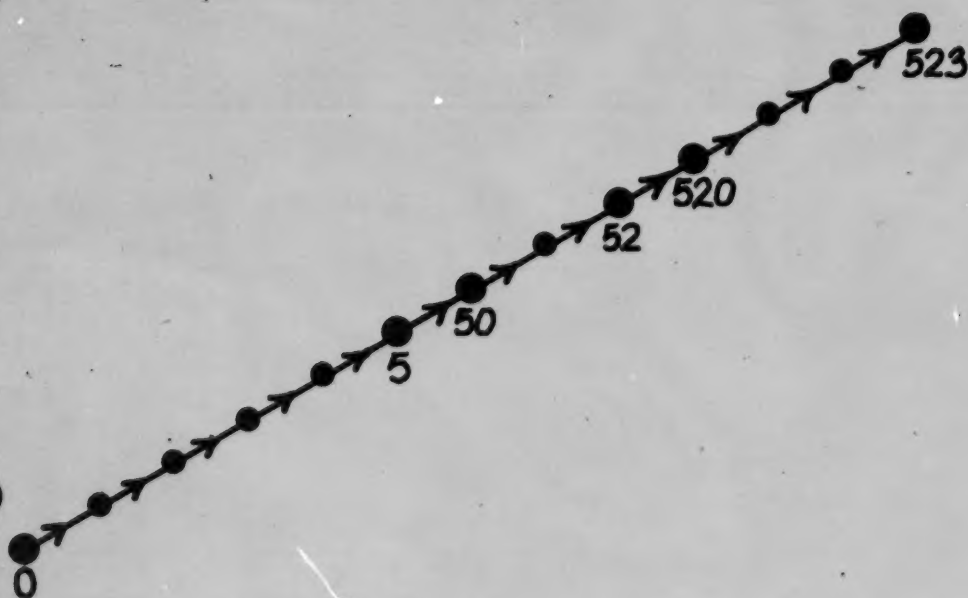
Draw them on the board:



T: What did you do next?

S: I used a red arrow.

With the help of your students, progressively complete the picture:



Choose any number on the board, for instance 271.

Ask the students to think how they could build a road starting at 0 and ending at 271 without actually drawing it.

After a while, have a student explain his or her solution.

S: First two blue arrows (from 0 to 2) and then a red arrow (to 20) and then 7 blue arrows (to 27) and then a red arrow (to 270) and finally a blue arrow (to 271).

As an optional exercise, you could ask students interested in this combinatorial work to make a list of all the possible ending numbers of roads starting from 0 and using exactly two red arrows and ten blue arrows.

Answer:

910	820	730	640	550
901	811	721	631	541
	802	712	622	532
		703	613	523
			604	514
				505

460	370	280	190
451	361	271	181
442	352	262	172
433	343	253	163
424	334	244	154
415	325	235	145
406	316	226	136
	307	217	127
		208	118
			109

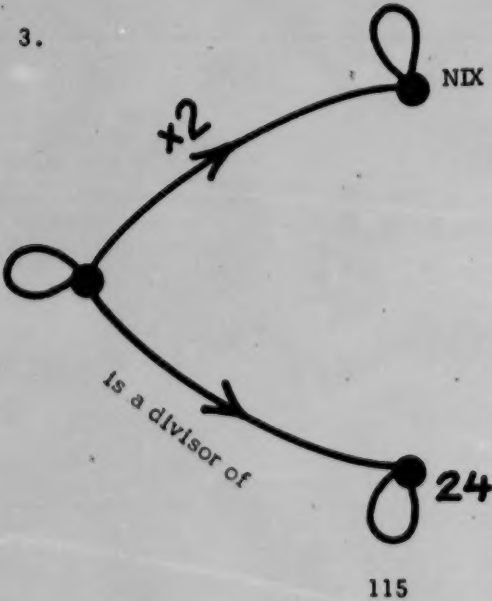
DETECTIVE STORIES

This chapter introduces a type of exciting pedagogical situation that will be used repeatedly in the CSMP Intermediate Grades Program. We refer to it as the "detective story approach." Some of the stories involve a set of clues given in sequence, which are couched in the various languages of the CSMP program. The object in each story is to determine one or two numbers, and the clues progressively diminish the possibilities until the identity of the number(s) is determined. Other detective stories present all necessary clues at once and the student works on his or her own, using the clues freely to find the number(s) in question.

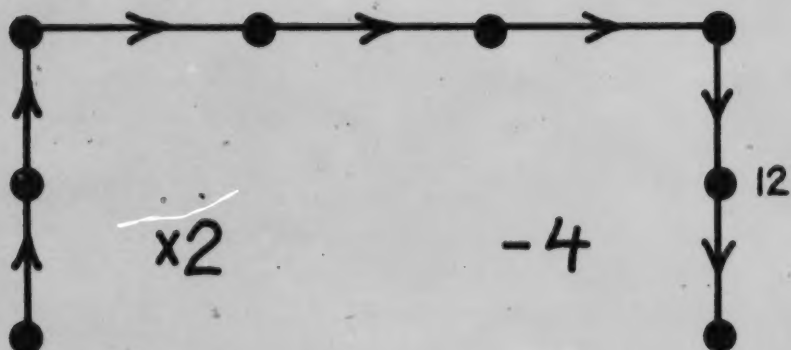
An example of the latter type of detective story is that of the number called "Nix." The clues given are:

1. Nix has exactly four positive divisors.
2. Nix can be put on the ones' board of the Minicomputer with exactly two checkers (positive or negative).

3.



4. Nix is in this arrow picture:



The stories are chosen so that each student, regardless of ability, can contribute at some stage to the determination of the number(s).

CAPSULE LESSON SUMMARIES

Section 5.1: Tell a detective story giving the clues in sequence.
(Worksheets N42, N43)

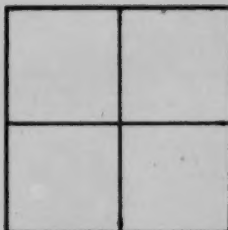
Section 5.2: Use the story-workbook, WHO ARE WE, to tell another detective story in which clues are given sequentially.

Section 5.3: Present two detective stories, the first about Nix and the second about Pif and Paf. In both stories give the clues all at once.
(Worksheets N44, N45)

Section 5.4: Present the detective story Nick and Nack, giving the clues sequentially.
(Worksheets N46, N47)

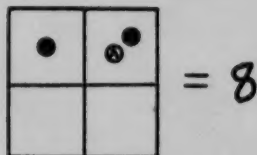
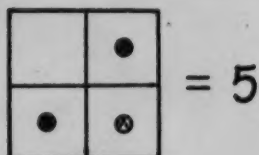
5.1 Find the Secret Number

Tell your students that in this lesson a "secret" number will be giving them some clues to help them find who it is. Display a demonstration Minicomputer board:



T: The secret number says, "I am on this board and I am using exactly two positive checkers and one negative checker. But you cannot see me because the checkers are invisible." Who could the secret number be?

Let some students show some of the possibilities on the Minicomputer. For instance,



T: What is the largest number our secret number could be?

6F

Answer:

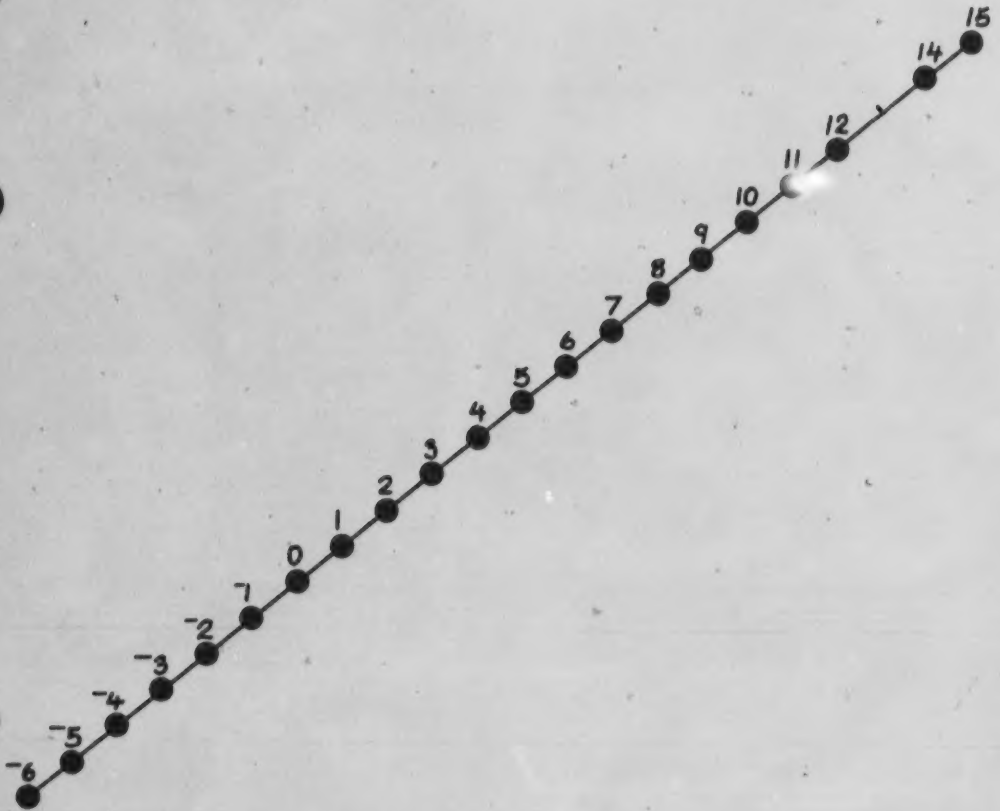
$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline & \ominus \\ \hline \end{array} = 15$$

T: What is the smallest number it could be?

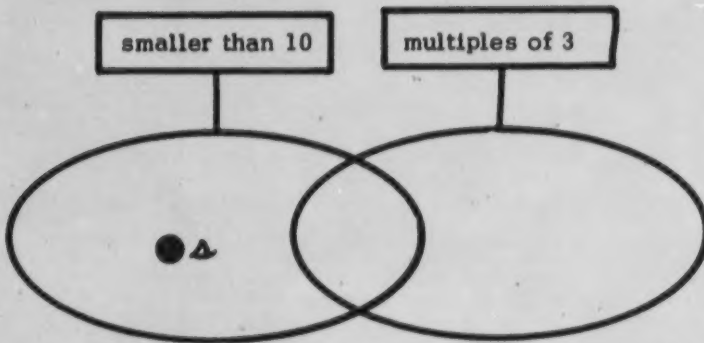
Answer:

$$\begin{array}{|c|c|} \hline \ominus & \\ \hline & \bullet \bullet \\ \hline \end{array} = -6$$

Give a copy of Worksheet N42 to each student and ask them to try to find all the numbers that the secret number could be. Tell them that there are 21 possibilities. When at least two or three students have found all the numbers, ask all the students to stop their individual work. Then check the answers together as a class. Start with 15, and then 14. The children should notice that 13 is missing but that from 12 to -6, we can put each number on the Minicomputer with two positive checkers and one negative checker. Put a dot for each possible number on a number line and ask the students to copy onto a sheet of paper the picture at the top of the next page.



Draw on the board:



T: "s" is for the secret number.

Is s a multiple of 3? (No)

Is s smaller than 10? (Yes)

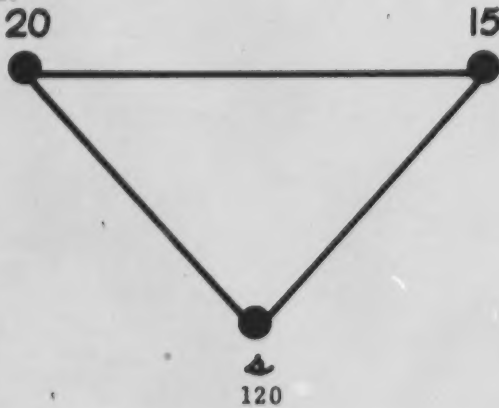
On your number line, cross out all the numbers that the secret number could not be.

Walk around the room and monitor the students' work. After a while, ask a student to cross out the impossible numbers on the board.



T: Now we know that the secret number is -5 or -4 or -2 or -1 or 1 or 2 or 4 or 5 or 7 or 8.

Draw on the board:



T: Do you remember the story of THE TALKATIVE NUMBERS?^{*}

Let a student explain that in this story two numbers could talk to each other (and their dots were joined by a red cord) if and only if one of the numbers was a multiple of the other. If not, their dots were joined by a blue cord.

T: Could the secret number be 8?

S: No, because 8 cannot talk to 20 and the cord between 20 and s is in red.

T: Could the secret number be 7?

S: No, because 7 cannot talk to 20 and the cord between s and 20 is in red.

T: Could the secret number be 5?

S: No, because 5 can talk to 15 and the cord between 15 and s is in blue.

T: Could the secret number be 4?

S: Yes, because 4 can talk to 20, and the cord between 20 and s is in red; and 4 cannot talk to 15, and the cord between 15 and s is in blue.

^{*} Veteran CSMP students studied this story-workbook in Part II of the Upper Primary Grades Program (third grade). Students who began the CSMP curriculum in the first semester of the fourth grade with CSMP Mathematics for the Intermediate Grades, Entry Module I studied THE TALKATIVE NUMBERS as part of that entry program. In either case, you may need to remind them of the story and the meaning of the cords.

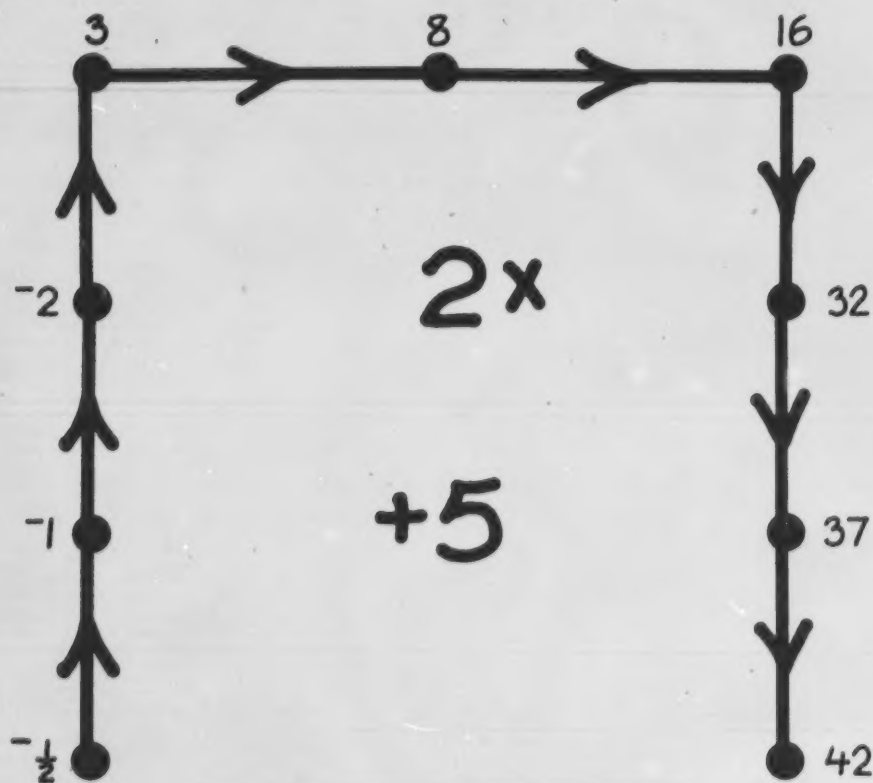
6F

Ask similar questions for 2 (yes); 1 (no); -2 (yes); -4 (yes) and -5 (no).

T: Now we know that the secret number is 2 or -2 or 4 or -4 .

Give a copy of Worksheet N43 to each student. Tell the children that the secret number is one of these dots and then ask them to find who it could be. While they are working by themselves, draw the picture of the worksheet on the board and after a while, let some students label the dots.

Solution:



Therefore the secret number is -2 .

Name _____

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N42 *

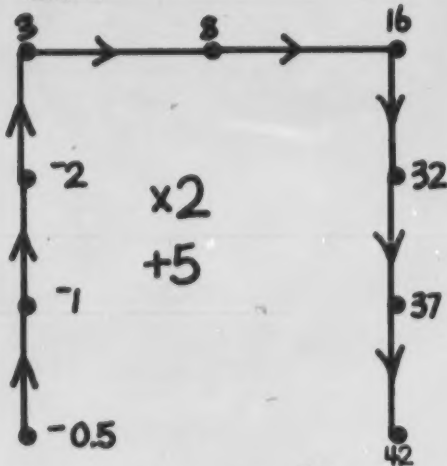
Write the 21 numbers that can be put on the ones' board with EXACTLY two positive checkers and one negative checker.

- 15	- 14	- 12
- 11	- 10	- 9
- 8	- 7	- 6
- 5	- 4	= 3
- 2	- 1	= 0
- 1	- 2	= 3
- 4	- 5	- 6

Name _____

Copyright © 1978 CEMREL, Inc.
N43 **

Label the dots:



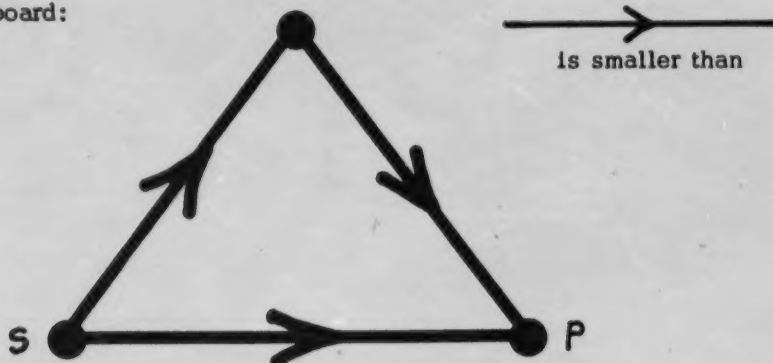
Now you know that the secret number is -2.

5.2 "Who Are We?"

Give a copy of the WHO ARE WE? story-workbook to each student.

T: By reading this detective story, you have to try to discover who the three secret numbers are. Read page 1 and answer the questions.

Draw on the board:



Starting from one of the dots and following one of the arrows, say:

T: This number is smaller than this one.
Did you find the smallest number in this picture?

S: s.

T: How do you know?

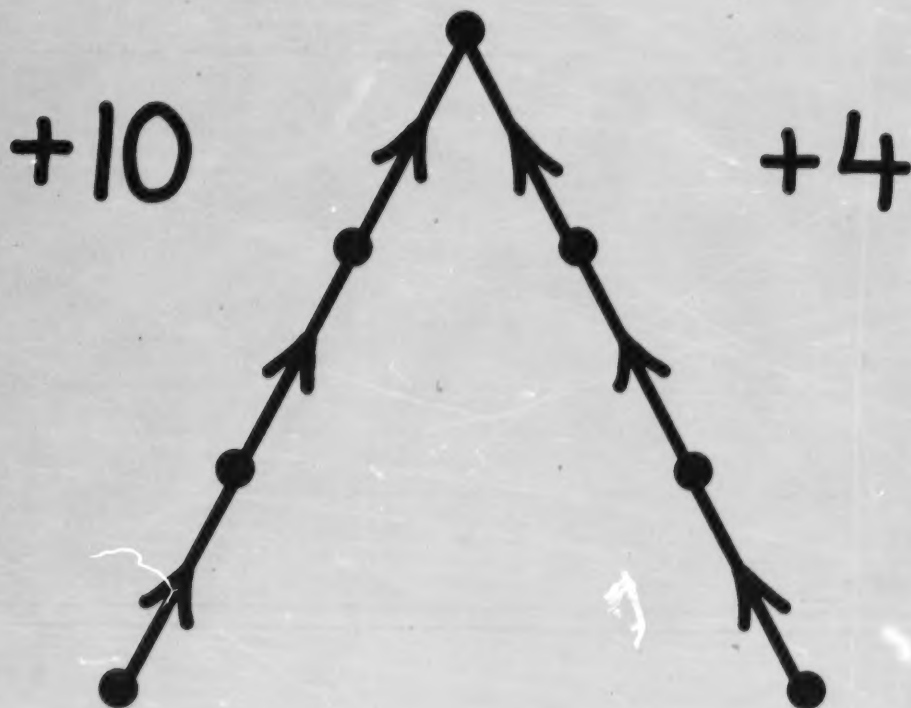
S: It is pointing to the two other numbers.
It is smaller than each of them.

T: Which is the largest number?

S: p, because the two other numbers are pointing to p.

T: Read pages 2-3 and try to solve the problems.

Draw the picture from page 3 on the board:



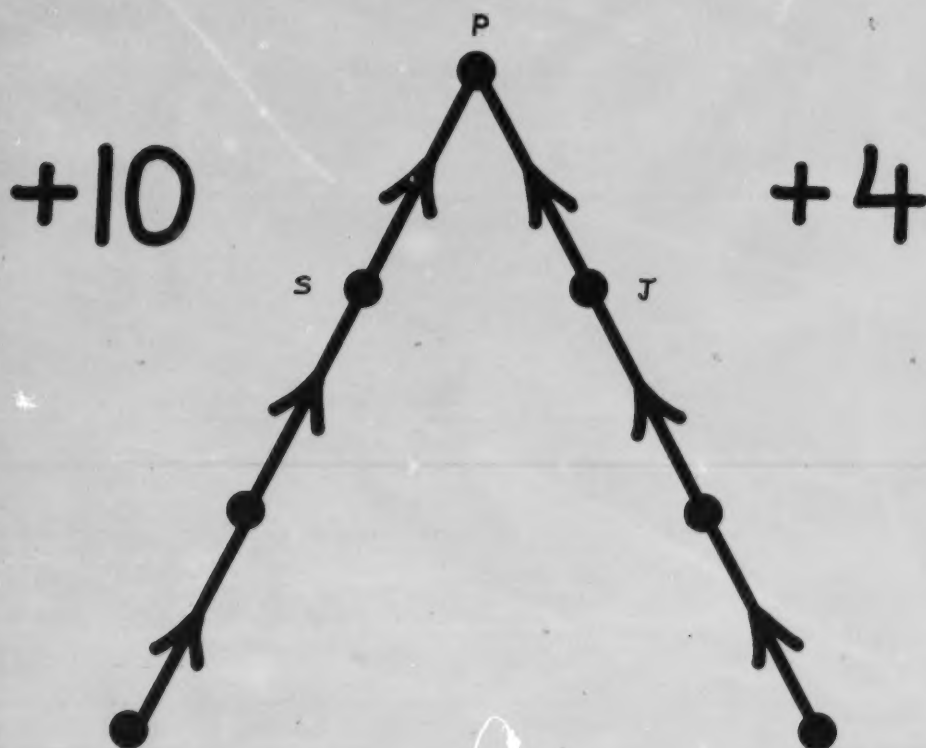
After a while, ask a student to point to the largest number in the picture (answer: the dot at the top); to point to the three largest numbers in the picture (answer: the three dots at the top).

T: Who is s? Who is p? Who is J?

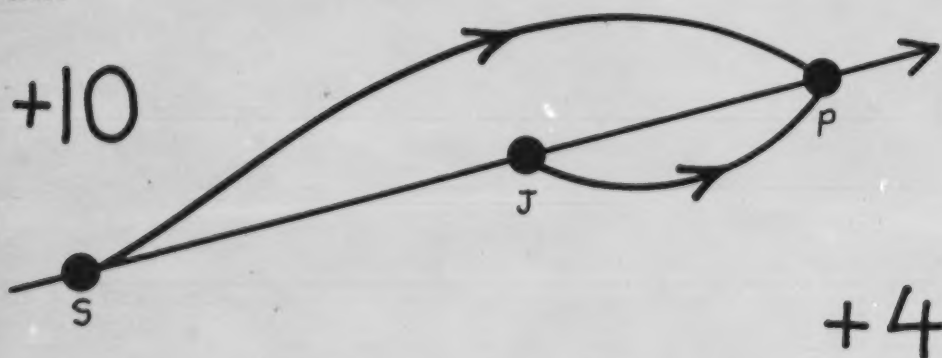
Don't forget that s is the smallest one of the three friends and p the largest one.

10F

Have a student write the names of the three numbers on the picture:



At this point, it would be helpful to draw the three numbers on the number line:



This picture should help the students to answer the questions on page 2:

p is 10 more than s

J is 4 less than p

J is 6 more than s

Have the students read pages 4 and 5. Draw the picture from page 5 on the board.

T: Did you find where J, s and p are?

Have a student label their dots on the board and explain his or her answer:



Have the students study pages 6-7. Draw the picture from page 7 on the board.

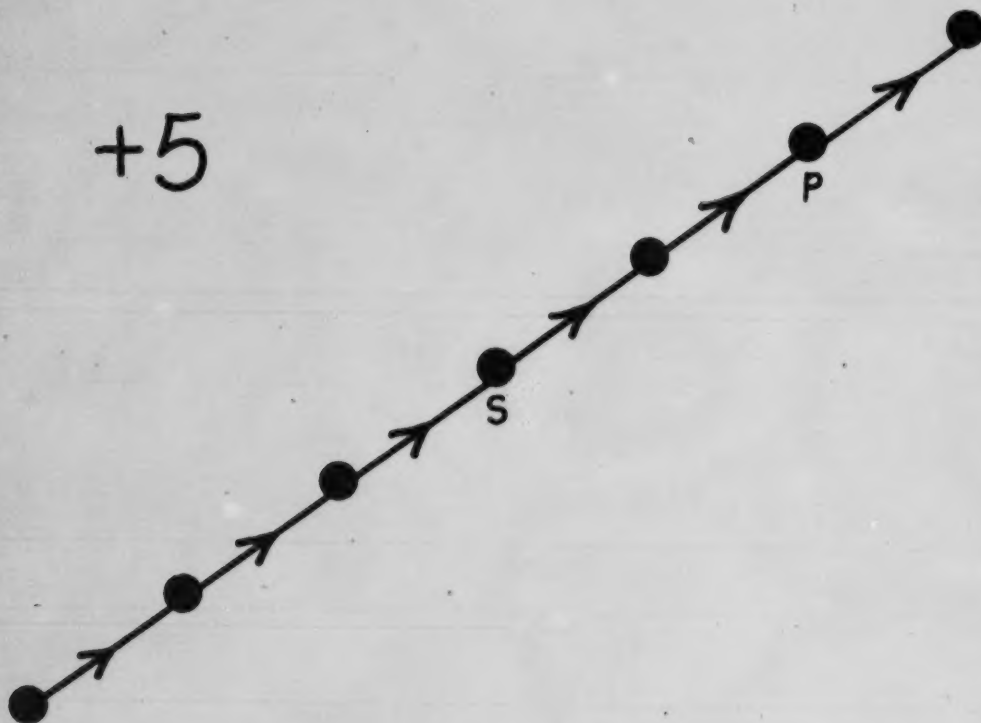
T: On which page is our friend J?

S: On page 6.

T: Write his name near the dot on page 6.

Did you find where s and p are on page 7?

Have a student label their dots on the board and explain his or her answer:



Have the students study pages 8-9.

Most of the students will probably agree that the red arrow could be for +10 or for "I am smaller than you" but cannot be for +6, nor for -6, nor for "I am larger than you."

T: Could the red arrow be for $2 \times$?
Can you think of two numbers for s and p who could play this game?

S: 10 and 20.

T: Can you think of two numbers for s and p who could not play this game?

S: 5 and 15; 20 and 30; 90 and 100; and so on.

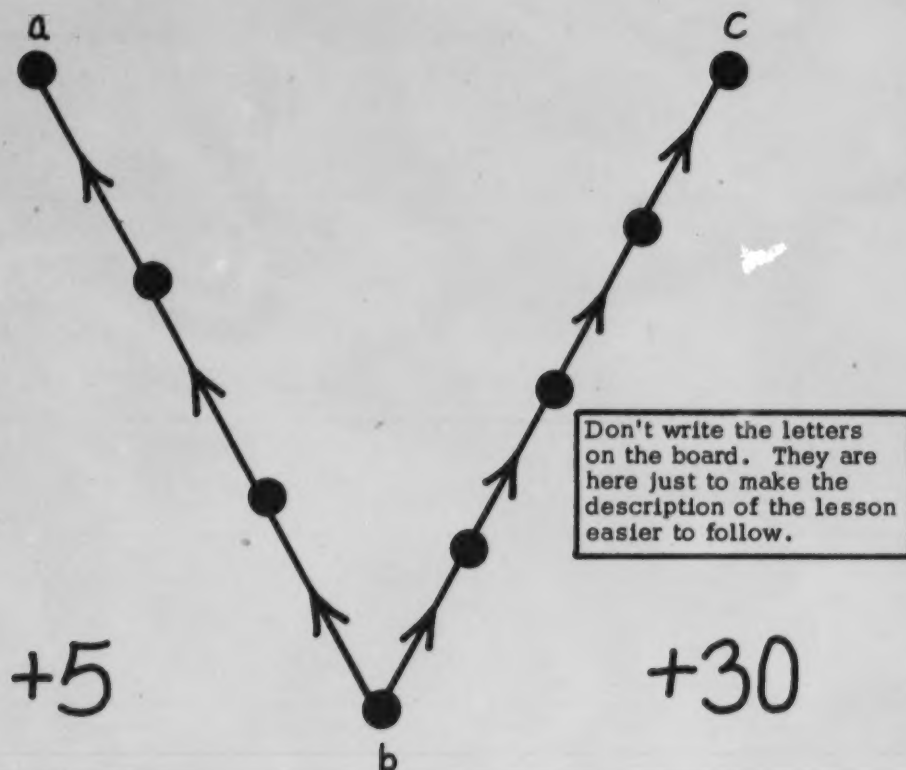
T: The question was: "Could the red arrow be for $2 \times$?" From the discussion we just had, how do you suggest we answer it?

S: We could say "maybe."

T: It's not a bad idea. To be more precise, we could write "Yes, if s and p were 10 and 20; no, if s and p were not 10 and 20."

Ask the students to read pages 10 and 11.

Draw the picture from page 11 on the board:



After a while ask:

T: How much larger is this number (point to a) than this one (point to b)?

S: 65.

T: How much larger is this number (point to c) than this one (point to b)?

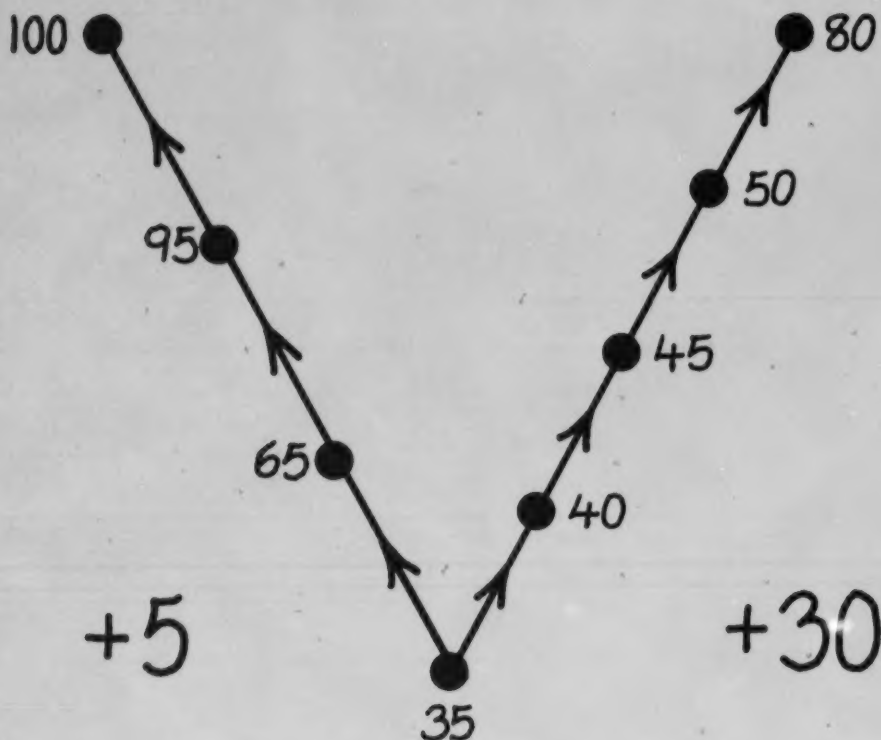
S: 45.

T: Where is the largest number in this picture?

S: At the top on the left (a).

Label this dot (point to a) "100" and then label all the other dots.

Have a student label the dots on the board:



T: What numbers could s and p be?

S: 35 and 45 or 40 and 50.

Before reading pages 12-13, ask the students to think of an odd number.

T: Add 7 to your number; add 7 again; add 7 once more. Is the final number even or odd?

10F

If some students answer "odd", check their calculations.

Do a similar exercise with an even number.

Then have the students solve the problems on pages 12-13.

T: Where is J in the picture on page 13?

S: At the dot labeled "me".

T: Is J even or odd?

S: Even.

T: And p?

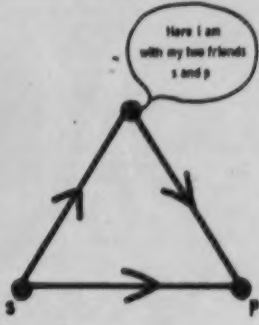
S: p is six more than J who is even.
So p is even.

S: And s is even too because s is 4 less than J.

T: Who are J, s and p?

Have the students write the answers on page 14 and explain why s is 40, p is 50 and J is 46.

"Who Are We?"



Here I am with my two friends s and p

Is smaller than

Who is smaller, s or p? s

Who is larger, s or p? p

I am between my two friends.

1

We are the three largest numbers in the picture on page 3.

Do you recognize us?

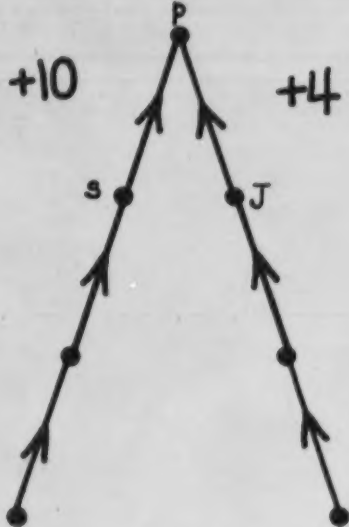
Write our names on the picture. I am called "J."

My friend p is 10 more than my friend s.

I am 4 less than my friend p.

I am 6 more than my friend s.

2



+10

+4

s

J

3

Don't forget that:

p is 4 more than J.

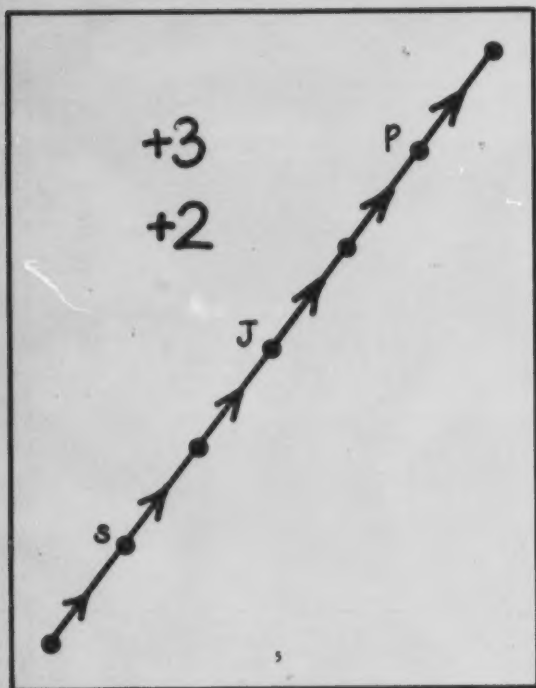
J is 6 more than s.

The three of us are in the picture on page 5.

Do you recognize us?

Write our names on the picture.

4



Here I am.

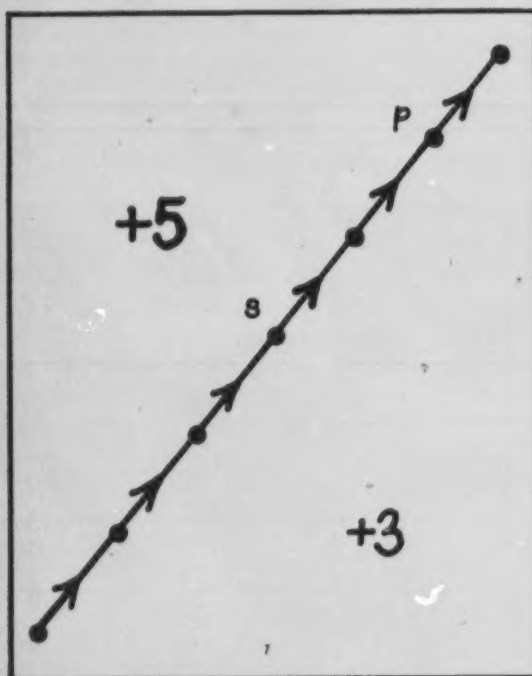
My two friends are in the picture on page 7.

Do you recognize them?

Write their names on the picture.

Don't forget that p is 10 more than s.

6



Here I am again

with my two friends

who are playing

together.

J

s P

8

Which game could my friends be playing?

Could their game be

I am larger than you?

No

I am smaller than you?

Yes

+ 6 ?

No

- 6 ?

No

2x ?

can't tell

+ 10 ?

Yes

- 10 ?

No

Here I am!
100 is the largest number in
the picture on page 11.
Do you see where 100 is?
Who are the other numbers?
Write their names on the picture.
My friends are two of
these numbers.

Now you know that

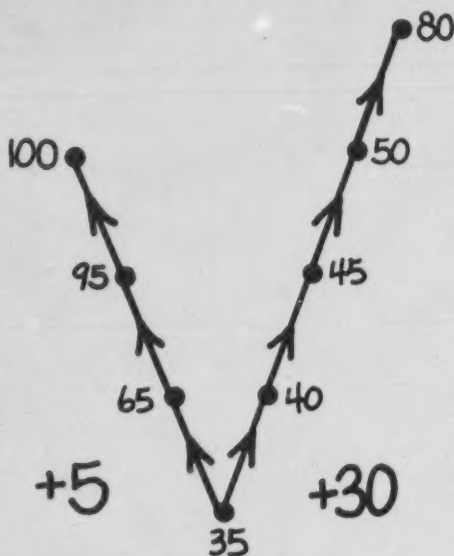
my friends are

35 ; 45

or

40 ; 50

10



11

Don't forget that:

p is 4 more than J.

J is 6 more than s.

Do you see me in the picture on page 13?

Am I even or odd?

even

Are my friends there too?

no

How do you know? Explain it below.

$S+6=J$, not $S+7$

$J+4=p$, not $J+7$

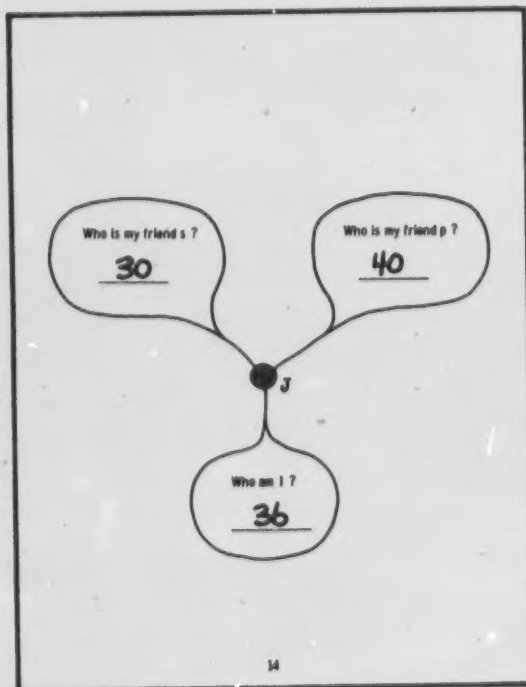
Is my friend s even or odd?

even

Is my friend p even or odd?

even

12

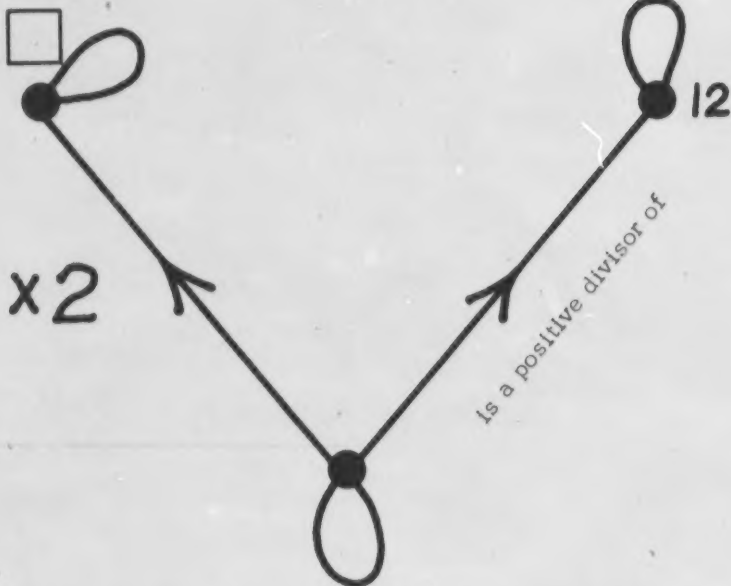


5.3 Nlx; Plf and Paf

Exercise 1 (introductory exercise):

Draw on the board:

+

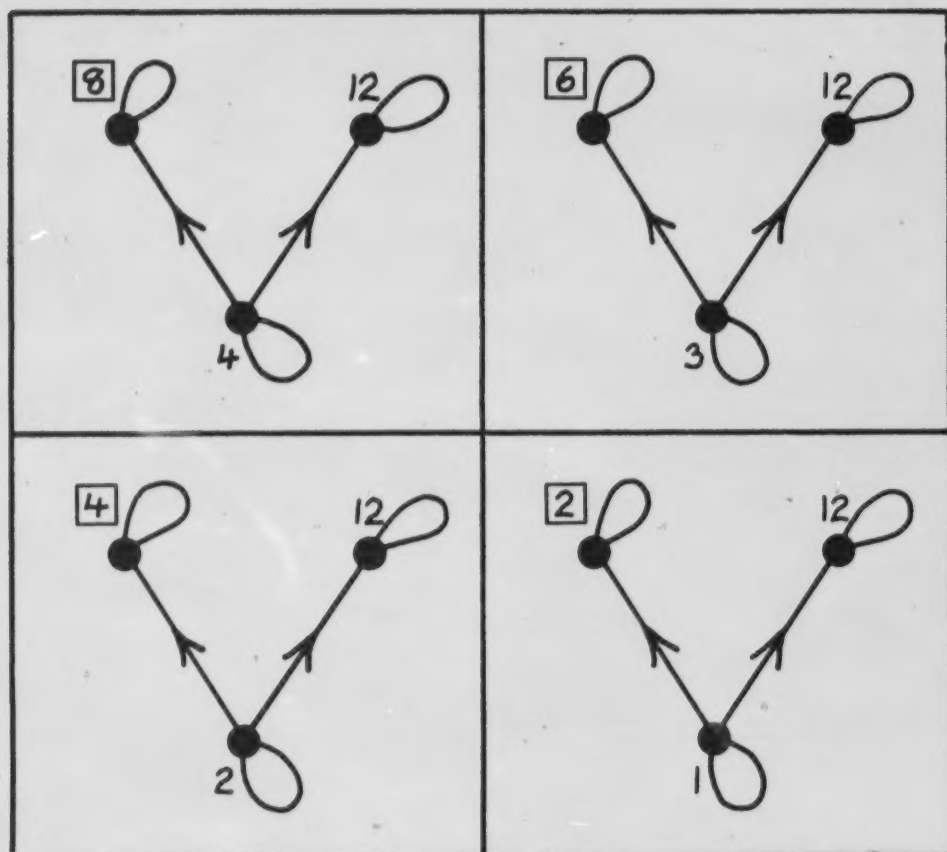


T: What number could be in the frame?

Tell the students to write some possible numbers on a sheet of paper.

After a while, let several of them explain their answers to the whole group.

Four solutions are possible:



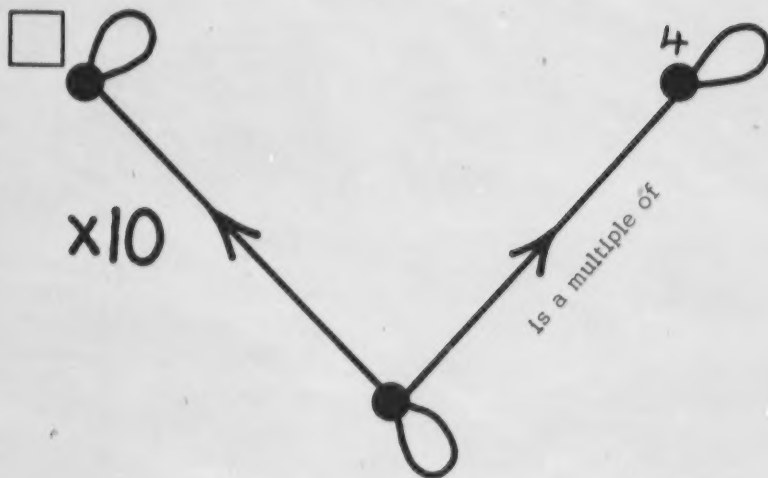
Erase "12" and replace it by "100". Ask a similar question. The possible numbers are: 50, 20, 10, 4, 2.

T: On your sheet of paper, write some numbers which have exactly four positive divisors.

Have the students share some of their answers.

(There is an infinite number of correct answers. Some are: 6, 8, 14, 15, 21, 35, 77, ... any positive integer which is the product of two distinct primes.)

Draw on the board:



T: What number could be in the frame? Write some possible answers on your sheet of paper.

After a while, ask several students to explain their answers to the whole group.

(There is an infinite number of answers: 80, 120, 160, 200, 240, 280, 320, 360, ... $10 \times$ any multiple of 4 which is not equal to 4.)

Exercise 2:

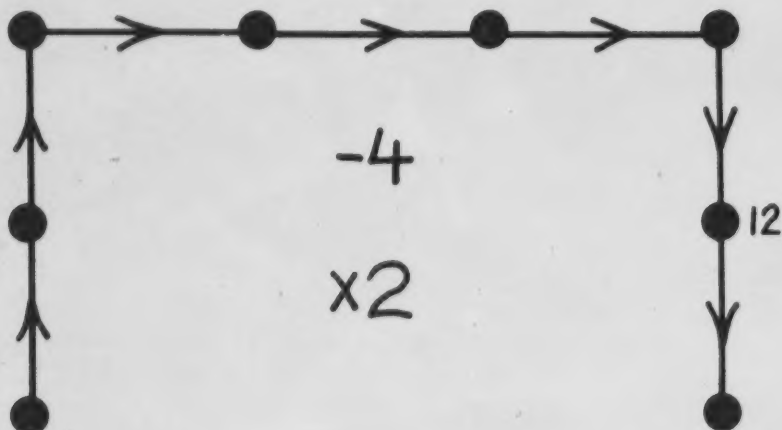
Give a copy of the "Nix" detective story to each student (Worksheet N44).

T: You have four clues to help you find which number Nix is. Read them carefully and use them in any order you wish.

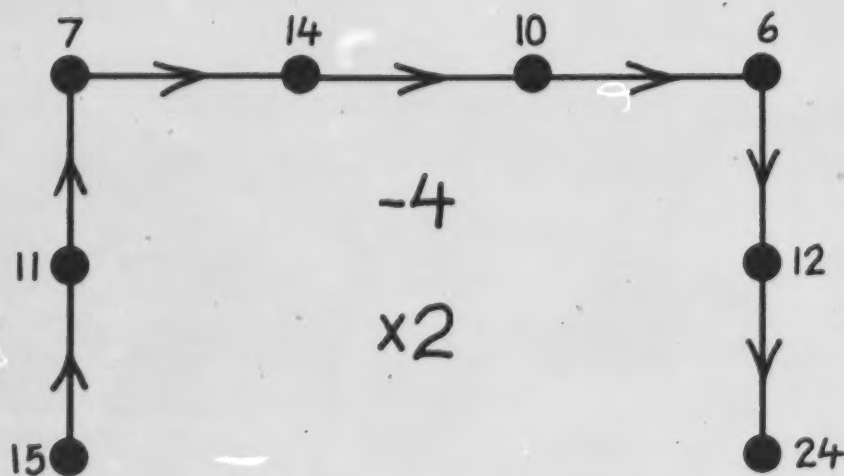
Have the students work on their own for a while.

As soon as a student has solved the problem, give him or her a copy of the "Pif and Paf" detective story (Worksheet N45).

After a while, draw on the board:



Have students label the dots:



T: Nix is one of these numbers. Could Nix be 24?

S: No, because 24 can't be put on the ones' board with only two checkers.

S: There's another reason why Nix can't be 24: 24 has more than four positive divisors.

Cross out "24" on the board.

T: Could Nix be 12?

S: No, because 12 has more than four positive divisors.

Cross out "12" on the board.

T: Could Nix be 6?

S: Yes, because 6 has exactly four positive divisors; 6 can be put on the ones' board with two checkers; $\frac{1}{2} \times 6 = 3$ and 3 is a divisor of 24.



T: Could Nix be 10?

S: No, because $\frac{1}{2} \times 10 = 5$ and 5 is not a divisor of 24.

Cross out "10" on the board.

T: Could Nix be 14?

S: No, because $\frac{1}{2} \times 14 = 7$ and 7 is not a divisor of 24. Also 14 cannot be put on the ones' board with only two checkers.

Cross out "14" on the board.

T: Could Nix be 7?

S: No, because 7 has only two positive divisors. Also $\frac{1}{2} \times 7$ is not a whole number and so is not a divisor of 24.

Cross out "7" on the board.

T: Could Nix be 11?

S: No, because 11 has only two positive divisors. Also $\frac{1}{2} \times 11$ is not a whole number and so is not a divisor of 24. Also 11 cannot be put on the ones' board with only two checkers.

Cross out "11" on the board.

T: Could Nix be 15?

S: No, because 15 cannot be put on the ones' board with only two checkers. Also $\frac{1}{2} \times 15$ is not a whole number and so is not a divisor of 24.

Cross out "15" on the board.

The students will conclude that Nix has to be 6.

Give a copy of the "Pif and Paf" detective story to each student (Worksheet N45) and have them work on their own for a while.

Write on the board:

_____ The positive divisors of 72 are _____

____; ____; ____; ____; ____; ____; ____; ____; ____; ____; ____; ____.

T: Try to find the twelve positive divisors of 72.

After a while, with the help of the class, write them on the board:

_____ The positive divisors of 72 are _____

1; 2; 3; 4; 6; 8; 9; 12; 18; 24; 36; 72.

Draw on the board:



T: Paf is how much larger than Pif?

Let several students whisper the answer (21 larger) in your ear and have one of them explain it to the whole class.

T: Now we know that Pif is one of the twelve positive divisors of 72 and that Paf is 21 more than Pif.

Help the students discover that

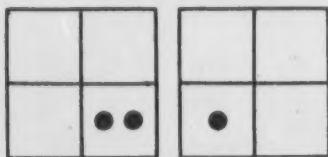
Paf is one of these numbers:

22; 23; 24; 25; 27; 29; 30; 33; 39; 45; 57; 93.

T: Could Paf be 22?

S: No, because 22 can be shown on the Minicomputer using exactly three checkers:

22 =



Cross out "22" on the board.

Ask a similar question successively for the next eleven numbers. The students should find that each number (with the exception of 57) can be written on the Minicomputer using exactly three checkers (positive or negative). For instance,

$$23 = \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline \bullet & \bullet \\ \hline \end{array}$$

$$24 = \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline \bullet \bullet & \\ \hline \end{array}$$

$$25 = \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \bullet \\ \hline & \bullet \\ \hline \end{array}$$

$$27 = \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array}$$

$$29 = \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array}$$

$$30 = \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \\ \hline \bullet & \\ \hline \end{array}$$

$$33 = \begin{array}{|c|c|} \hline & \bullet \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array}$$

$$39 = \begin{array}{|c|c|} \hline & \\ \hline \bullet \bullet & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \bullet \\ \hline \end{array}$$

$$45 = \begin{array}{|c|c|} \hline & \bullet \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \bullet \\ \hline & \bullet \\ \hline \end{array}$$

$$43 = \begin{array}{|c|c|} \hline & \\ \hline & \bullet \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array}$$

Therefore, the students will conclude that Paf is 57 and Pif is $57 - 21 = 36$.

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Name _____ N44 *

Nix

Nix has exactly four positive divisors.
(15, 14, 10, 6)
Nix is on this Minicomputer board using exactly two checkers (positive or negative). You cannot see the checkers because they are invisible.

(10, 6)

7

11

15

14

$\times 2$

$- 4$

18

12

24

Nix is in this picture.

$\times 2$

3

is a divisor of

24

Nix = 6

N44

The following sequence is one strategy for solving this problem:

- 1) Complete the arrow picture at the bottom of the page.
- 2) The only numbers in the arrow picture with exactly four positive divisors are 15, 14, 10 and 6. So Nix is one of these.
- 3) Among these four numbers only 10 and 6 can be put on the Minicomputer with exactly two checkers.
- 4) The divisors of 34 are: 1, 2, 17, 34. The double of one of these numbers is Nix according to the arrow picture in the upper right hand part of the page. But we know Nix is 6 or 10. Since $2 \times 3 = 6$ and 6 is not a divisor of 34, Nix must be 6.

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Name _____ N45 **

Pif and Paf

Refer to the student worksheet for the problem.

The following sequence is one strategy for solving this problem:

- 1) From the second clue we know Pif is one of these numbers: 1; 2; 3; 4; 6; 8; 9; 12; 16; 24; 36; 72.
- 2) From the first clue we know Paf is 21 larger than Pif. So Paf is one of these numbers: 22; 23; 24; 25; 27; 28; 30; 33; 39; 45; 57; 93.
- 3) Among the possibilities for Paf only 57 cannot be put on the Minicomputer with exactly three checkers (positive or negative). One way of writing each of the other numbers is shown below:

$\begin{bmatrix} \square & \square & \square \\ \circ & \square & \square \end{bmatrix} = 22$

$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} = 23$

$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} = 24$

$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} = 25$

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$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} = 33$

$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} = 39$

$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} = 45$

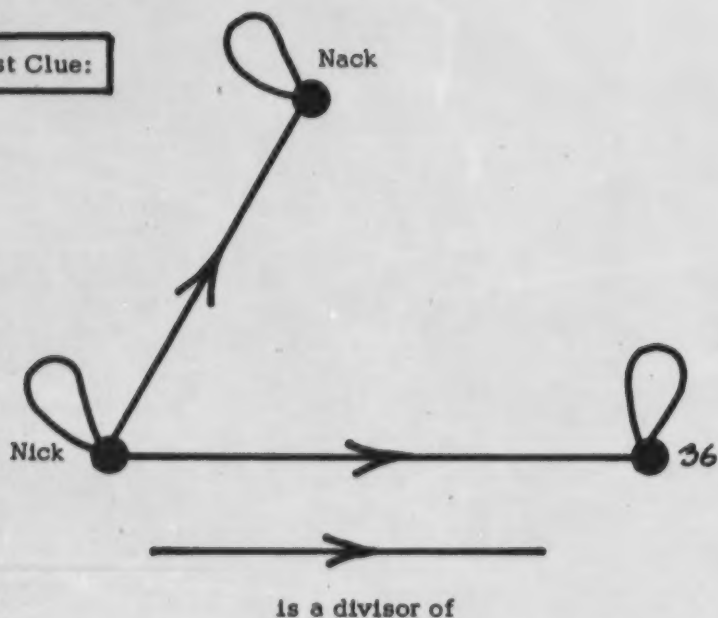
$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} = 93$

4) So Paf is 57 and Pif is 36.

5.4 Nick and Nack

Draw on the board:

First Clue:



T: Nick and Nack are two secret positive integers.
What do you know about Nick?

S: Nick is a divisor of 36 but different from 36.

S: Nick is a divisor of Nack.

T: What do you know about Nack?

S: Nack is a multiple of Nick.

T: Could Nick be 5?

S: No, because 5 is not a divisor of 36.

T: What numbers could Nick be?

S: 1; 2; 3; 4; 6; 9; 12; 18.

T: Could Nack be 7?

S: Yes, but in this case, Nick would have to be 1.

T: Which is the larger number, Nick or Nack?

S: Nack, because he is a multiple of Nick and both are positive integers.

In one corner of the board, write:

Nick is 1 or 2 or 3 or 4 or 6 or 9 or 12 or 18.

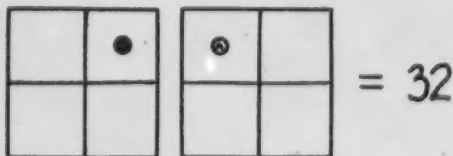
Nack is larger than Nick.

T: I shall give you a second clue: It is impossible for Nack to be put on the Minicomputer using exactly two checkers (positive or negative). Could Nack be 14?

S: 14 can be put on the Minicomputer with two (positive) checkers. So Nack cannot be 14.

T: Could Nack be 32?

Have a student put 32 with two checkers and conclude that Nack cannot be 32.



$$= 32$$

T: Could Nack be 33?

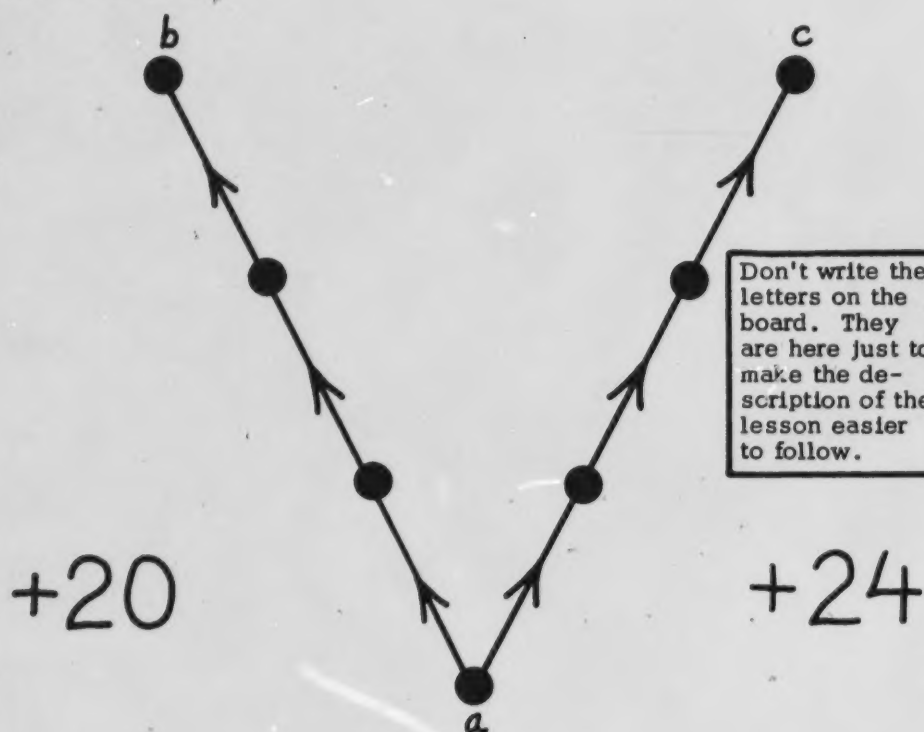
S: Yes, because it's impossible to put 33 on the Minicomputer using only two checkers.

Ask a similar question for some more numbers; for instance, 78, 96, 103 and 980. (Nack could be 103 but not 78, nor 96 nor 980.)

Give Worksheet N46 to each student and ask them to solve these problems on their own. After a while, check the answers in a collective discussion.

Give Worksheet N47 to each student and ask them to try to find where Nick and Nack are in this picture.

Draw on the board:



T: Circle the smallest number in your picture.

Ask a student to circle it (a) on the board.

T: What is the largest number in the picture?

S: The one at the top on the right (c).

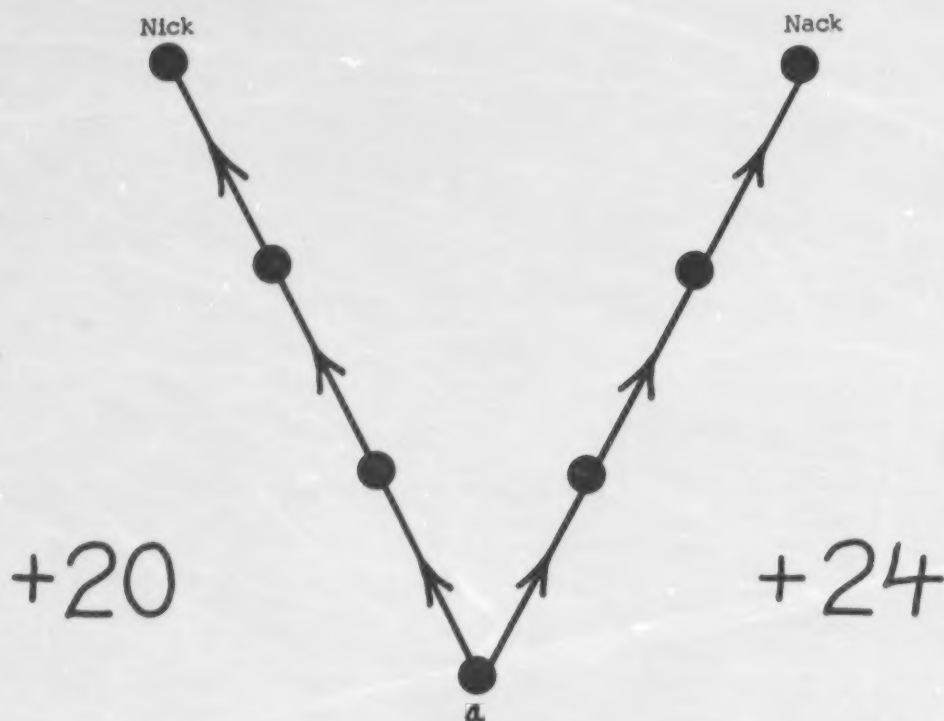
T: How do you know? Come to the board and explain.

S: This number (pointing to c) is 72 more than this number (pointing to a) and this number (pointing to b) is only 60 more than this number (pointing to a).

T: Which are the two largest numbers in this picture?

S: The two at the top (b and c).

T: Which one is Nick and which one is Nack? Write their names on the picture. (Answer: Nick is b and Nack is c because Nack is larger than Nick.)



T: By looking at this picture, you can find how much larger Nack is than Nick. Write your answers on a sheet of paper. (Call on someone to give the answer orally.)

S: Nack is 12 more than Nick.

T: How do you know? Come to the board and explain.

S: Nack is 72 more than this number (pointing to a) and Nick is only 60 more than this number (pointing to a). So Nack is 12 more than Nick.

T: Do you remember the numbers that Nick could be?

Point to the corner of the board where they are written.

T: What do you think about Nack?

S: Nack is 12 more than Nick. So Nack has to be one of these numbers: 13; 14; 15; 16; 18; 21; 24; 30.

T: Haven't you forgotten something else about Nack?

S: It is impossible to show Nack on the Minicomputer using only two checkers.

Ask the students to cross out the numbers that Nack couldn't be from the above list. (Answer: 14, 16, 18, 21, 24 and 30 have to be crossed out because each of these numbers can be put on the Minicomputer with two checkers.)

T: What do you know about Nack?

S: Nack is 13 or 15.

T: What do you know about Nick?

S: Nick is 12 less than Nack. So, Nick is 1 or 3.

Write on the board:

Nack has exactly four positive divisors.

T: Write who Nack is and who Nick is on your sheet of paper. (Answer: Nack is 15 because 15 has four positive divisors and 13 has only two. Nick is 3 because $3 + 12 = 15$.)

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Name _____

N46 *

Could Nick be 77	<u>No</u>	<table border="1"><tr><td></td><td></td><td></td><td></td><td>●</td></tr><tr><td></td><td></td><td></td><td></td><td>●</td></tr></table>					●					●
				●								
				●								
Could Nick be 157	<u>Yes</u>	<table border="1"><tr><td></td><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>										
Could Nick be 367	<u>No</u>	<table border="1"><tr><td></td><td></td><td></td><td>●</td><td>●</td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>				●	●					
			●	●								
Could Nick be 187	<u>No</u>	<table border="1"><tr><td></td><td></td><td>●</td><td></td><td>●</td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>			●		●					
		●		●								
Could Nick be 3097	<u>No</u>	<table border="1"><tr><td></td><td></td><td>●</td><td>●</td><td></td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>			●	●						
		●	●									
Could Nick be 3407	<u>Yes</u>	<table border="1"><tr><td></td><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>										
Could Nick be 9907	<u>Yes</u>	<table border="1"><tr><td></td><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>										
Could Nick be 1807	<u>No</u>	<table border="1"><tr><td></td><td></td><td>●</td><td></td><td>●</td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>			●		●					
		●		●								
Could Nick be 7, 9897	<u>No</u>	<table border="1"><tr><td>●</td><td></td><td></td><td>●</td><td></td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>	●			●						
●			●									
Could Nick be 7, 9397	<u>Yes</u>	<table border="1"><tr><td></td><td></td><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>										

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Name _____

N47 *

Third clue:

Nick and Nick are the largest two numbers in this picture:

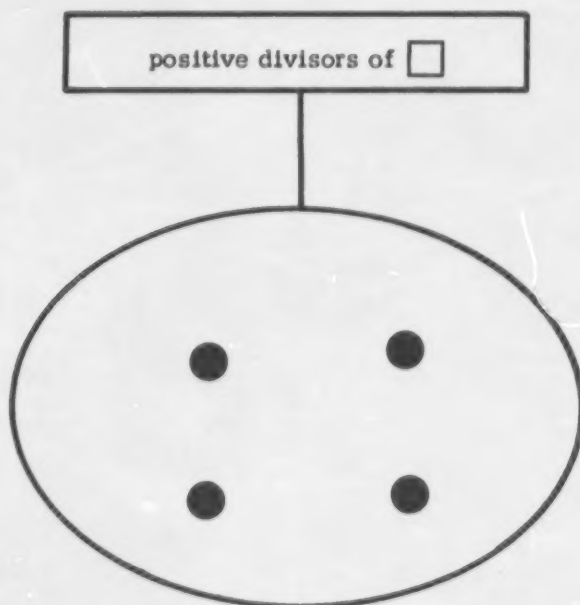
N47

(From first clue, Nick is:
1; 2; 3; 4; 6; 9; 12; or 18.
From second clue, Nick is:
13; 14; 15; 16; 18; 21; 24;
30.)

DIVISORS

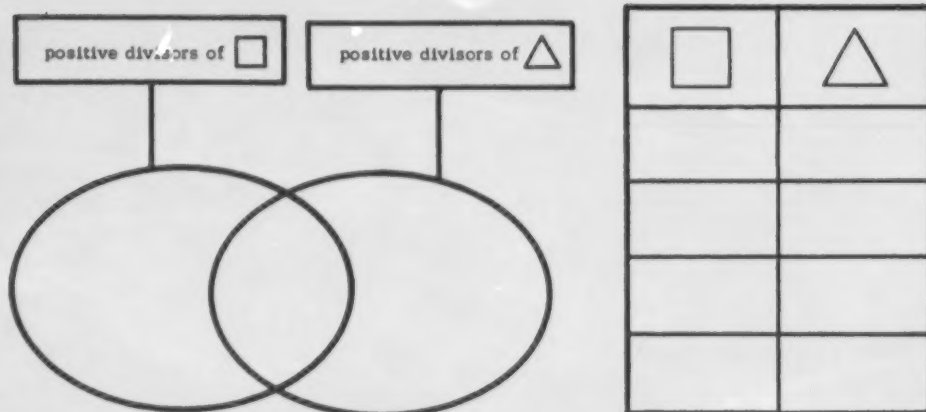
The relations "I am a multiple of you" and "I am a divisor of you" (on the set of positive integers) which are converses of each other, are presented in a variety of situations utilizing the languages of arrows and strings. In particular, the language of strings is used as visual support for the definition of positive prime integers (those with exactly two positive divisors).

Questions such as: "Given the string picture



what number could go in the box?" get at the notion of numbers which have exactly four positive divisors.

Another situation presented is: "Given the string picture



what numbers could go in the box and in the triangle?"

The situations presented in this chapter represent a first investigation of the basic building blocks of the integers, namely, the primes, and the representation of each integer as the product of primes.

CAPSULE LESSON SUMMARIES

Section 6.1: Examine the relations "is a multiple of" and "is a divisor of" in concrete situations, utilizing arrow diagrams and the language of strings. With the aid of arrow diagrams, it becomes clear that these two relations are converses of each other. Define positive prime integers using string pictures.

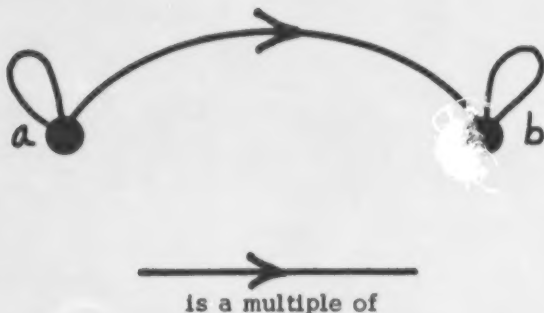
Section 6.2: Explore the "is a divisor of" and "is a multiple of" relations in a variety of situations.

6.1 The Relation, "Is A Divisor Of" #1

At the beginning of this lesson tell your students that throughout this lesson, all the dots you draw will be for positive integers.

Exercise 1:

Draw on the board:

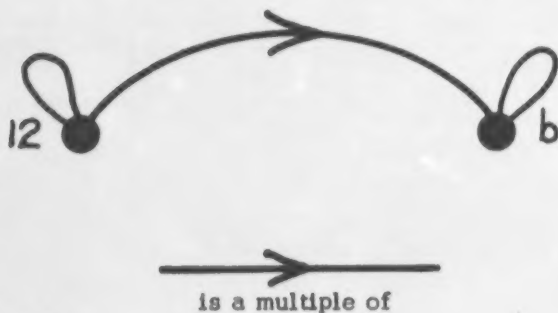


Don't write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: This number (point to a) is a multiple of this number (point to b).
Which numbers could these dots be for?

Have the students give several solutions.

Write "12" on the picture as follows:



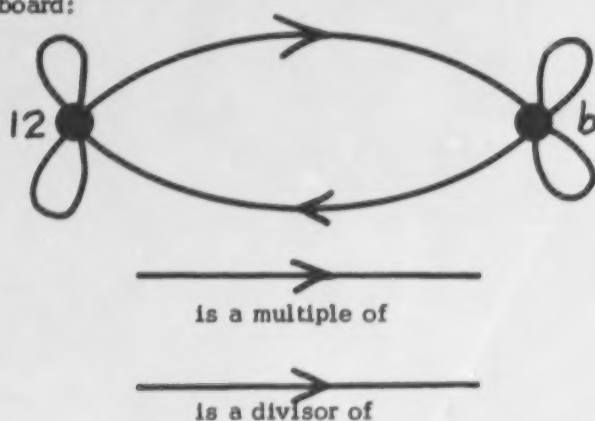
T: Which numbers could this dot (point to b) be for?
Write all the possible numbers on a sheet of paper.

S: 1; 2; 3; 4; 6.

T: Yes, 12 is a multiple of 1; 2; 3; 4; 6 ... and 12. This last fact is indicated by the red loop. Remember our convention: only one dot for each number.

T: I am going to introduce a new word. See if you can tell me what it means by looking at the picture.

Draw on the board:



Listen to the comments of the students and say:

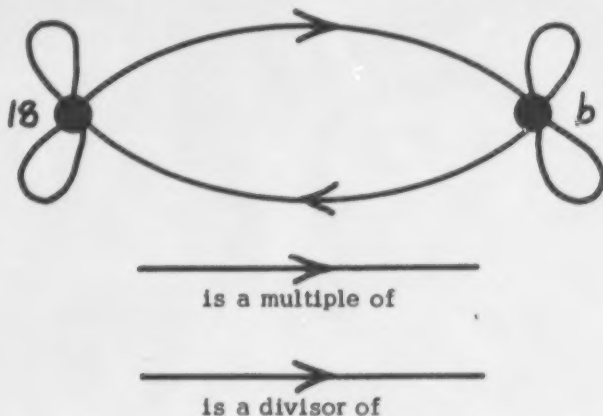
T: This number (point to b) is a positive divisor of 12.
What are the positive divisors of 12?

S: 1; 2; 3; 4; 6 and 12.

T: Do you understand why these numbers are called divisors of 12?
(Answer: a positive divisor of 12 is a positive integer that divides 12 evenly, with no remainder)

Accept any answer which conveys this idea and help the students express it more clearly if necessary. If no one can express the idea, tell the students yourself.

Erase "12" and write "18" on the picture as follows:

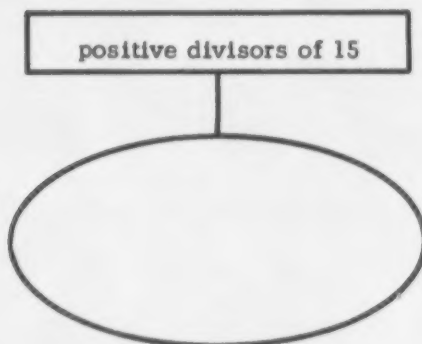


T: 18 is a multiple of this number (point to b). This number (point to b) is a divisor of 18. What are the positive divisors of 18? Write them on a sheet of paper. (Answer: 1; 2; 3; 6; 9; 18)

Erase "18" and write "30" in its place. Ask a similar question.

Exercise 2:

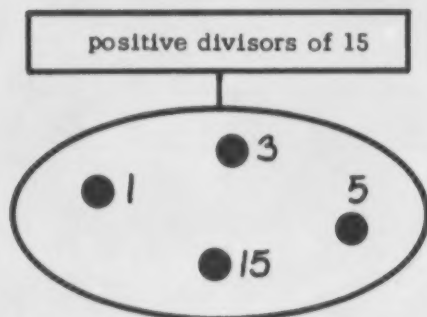
Draw on the board:



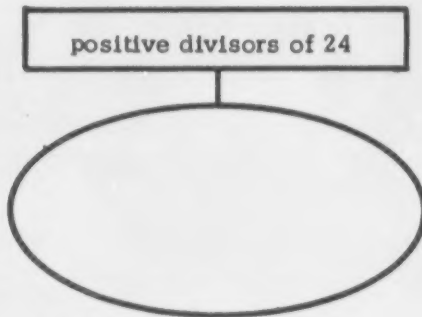
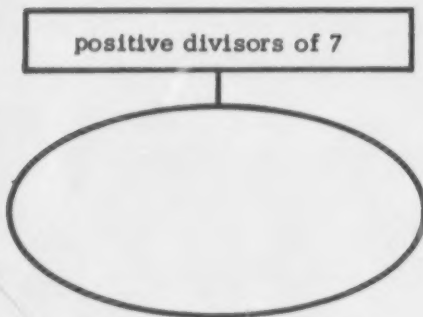
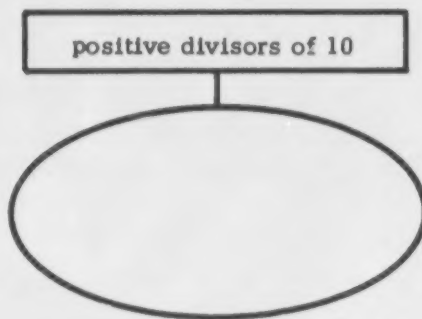
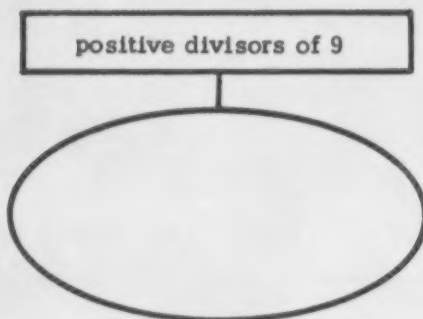
T: How many dots must you draw to include all the positive divisors of 15 in the red string?

S: Four dots.

Label each dot:



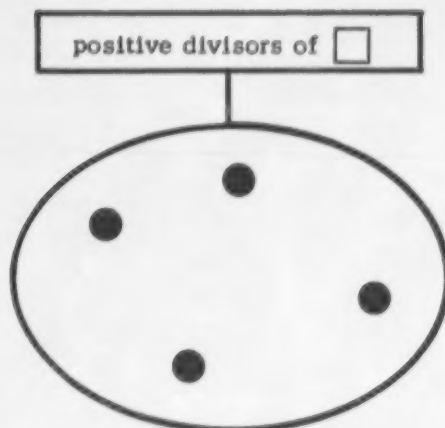
Do a similar exercise for:



If you have some fast workers, draw some more strings and have them do similar exercises.

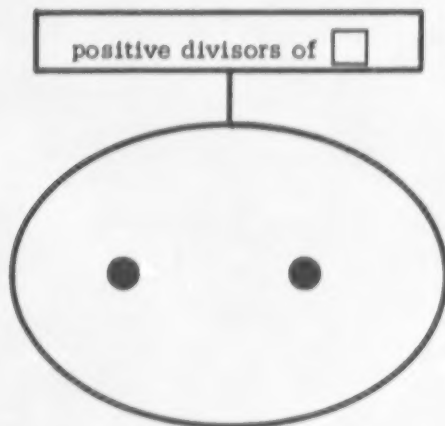
Exercise 3:

Draw on the board:



- T: There are exactly four positive integers inside the red string. What number could be in the little box? There are many possible answers. Write some of them on a sheet of paper. (Answer: 8; 10; 14; 15; 21; ... are all numbers which have exactly four positive divisors.)

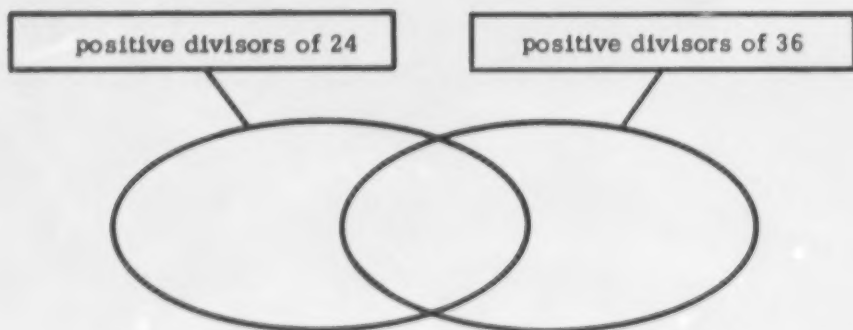
Draw on the board:



- T: There are exactly two numbers inside the red string.
Which number could be in the little box? Many answers are possible.
Write some of them on a sheet of paper. (Answer: 2; 3; 5; 7; 11; 13;
... are all numbers which have exactly two positive divisors.)
- T: The integers which have exactly two positive divisors are called
prime numbers. Write some more prime numbers on your sheet of
paper. (Answer: Some other prime numbers are 17; 19; 23; 29; and
so on.)

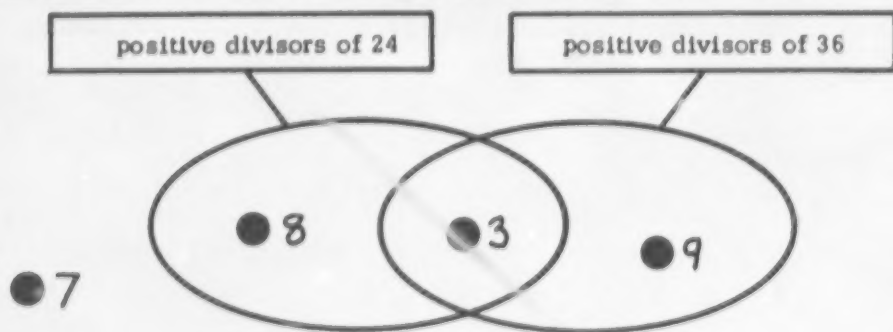
Exercise 4:

Draw on the board:

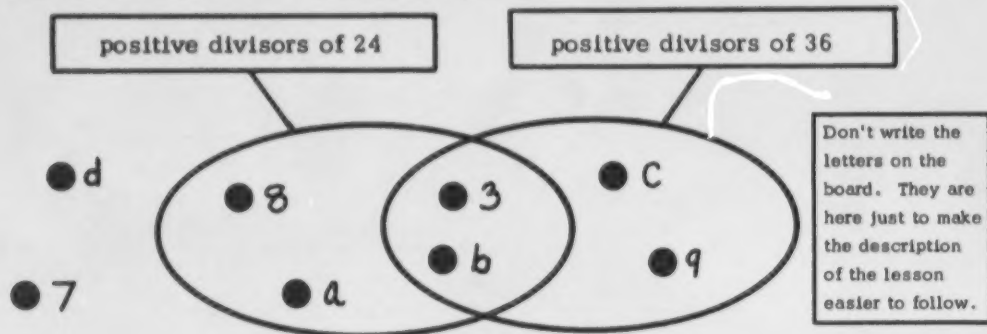


- T: Copy this picture on a sheet of paper and put in a dot for 8; 7; 3 and 9.

After a while, ask some students to put the four dots on the board:



Draw a new dot in each region of the picture and ask:



T: Which number could this dot (point to a) be for?

S: 24.

T: Which number could this dot (point to b) be for?

S: 1 or 2 or 4 or 6 or 12.

T: Which number could this dot (point to c) be for?

S: 18 or 36.

T: Which number could this dot (point to d) be for?

S: Any number which is neither a positive divisor of 24 nor a positive divisor of 36. (For example, 11, 0, -2 , $\frac{1}{2}$, 8.5, ...)

S: It could also be for something other than numbers like my parents' car or this pencil or you.

T: In the beginning of this lesson we said all dots were for positive integers but for the dots outside both strings, I'll accept all these answers like 0, -2 , $\frac{1}{2}$, 8.5, this pencil, your pet monkey, and so on. They are very clever answers for these dots if I remove the restriction that I put on at the beginning of the lesson.

6.2 The Relation, "Is A Divisor Of" #2

At the beginning of this lesson tell your students that throughout this lesson, all the dots you draw will be for positive integers.

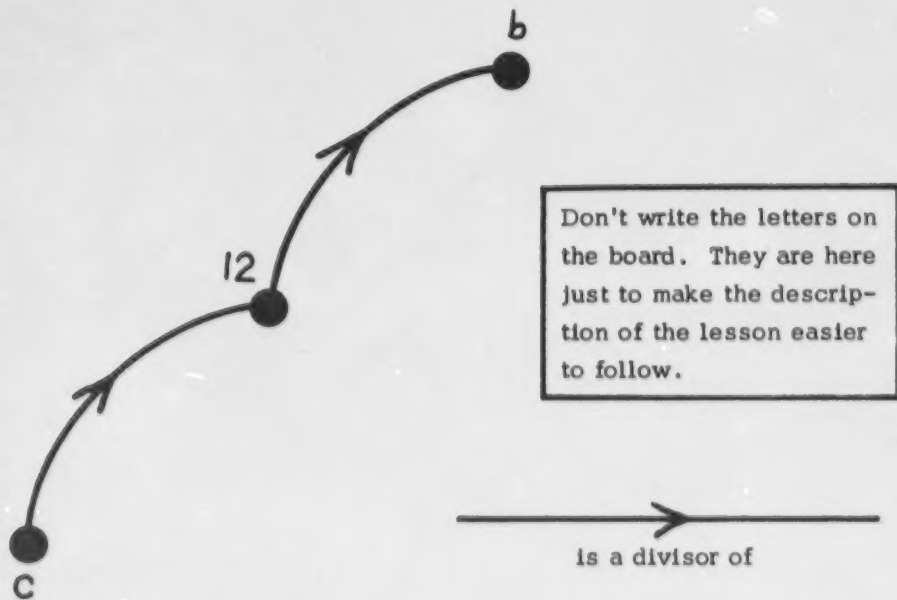
Exercise 1:

T: What are the positive divisors of 14? Write them on a sheet of paper.
(Answer: 1; 2; 7; 14)

Ask a similar question for some other numbers; for instance, 9; 15; 11; 21; 24.

Exercise 2:

Draw on the board:



Point to c and ask:

T: Which number could this dot be for?

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S: 1; 2; 3; 4; 6.

T: This dot (point to c) could be for any divisor of 12 with the exception of 12.

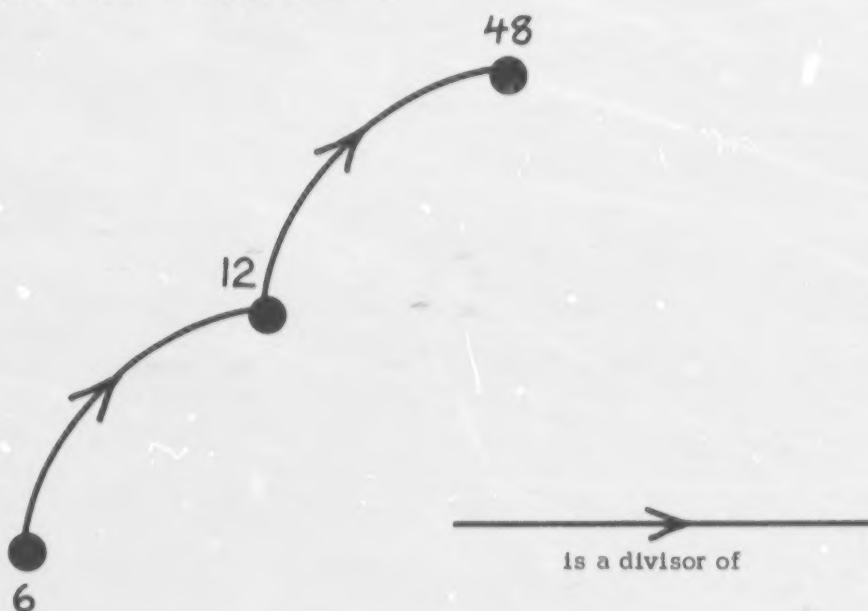
Which number could the other dot (point to b) be for?

Write some possible numbers on a sheet of paper. (Answer: 24; 36; 48; 60; 72; 84; and so on.)

What do you notice about these numbers?

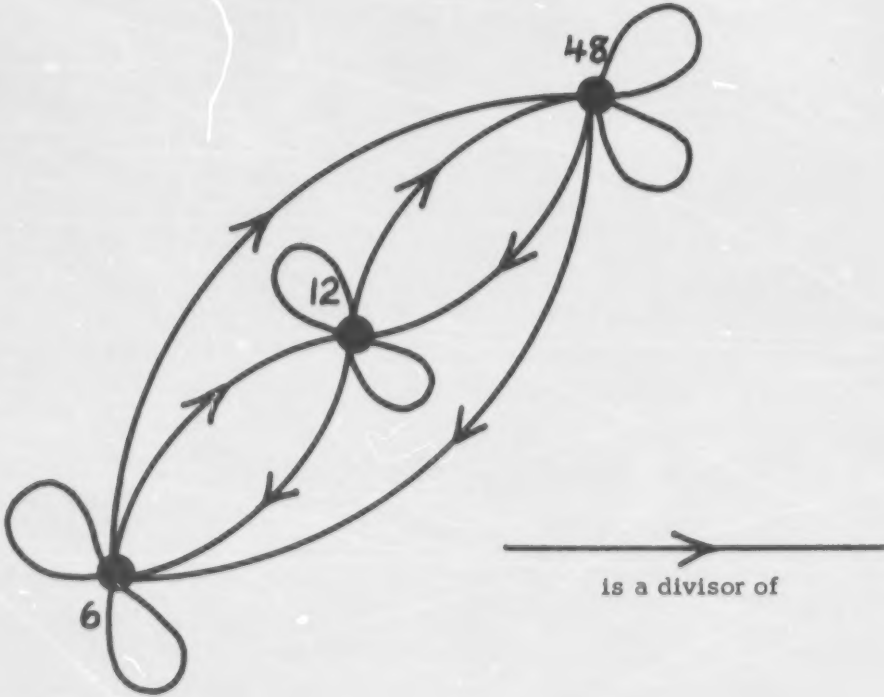
S: They are multiples of 12.

Ask a student to write a number for each dot on the board. Let's suppose that he or she chooses 6 and 48.



T: Some blue arrows and loops are missing in this picture. Can you trace them with your finger?

Have students take turns and draw the four missing blue arrows and loops. Then use red to draw the return arrows.

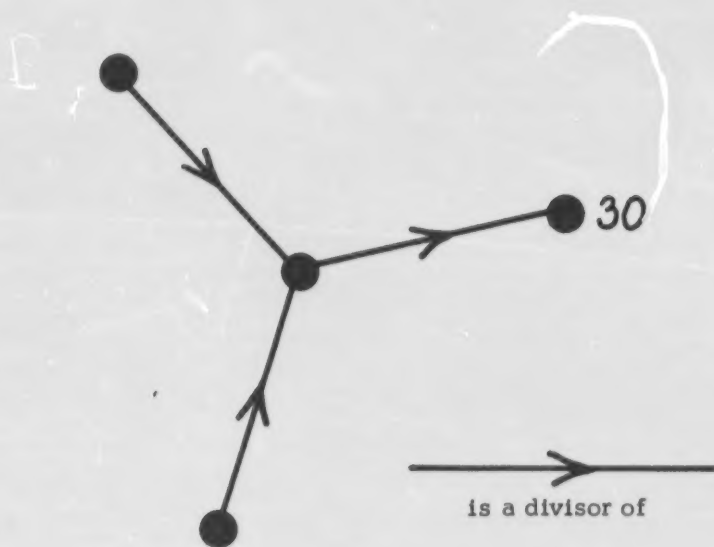


T: What could the red arrows be for?

S: Is a multiple of.

Exercise 3:

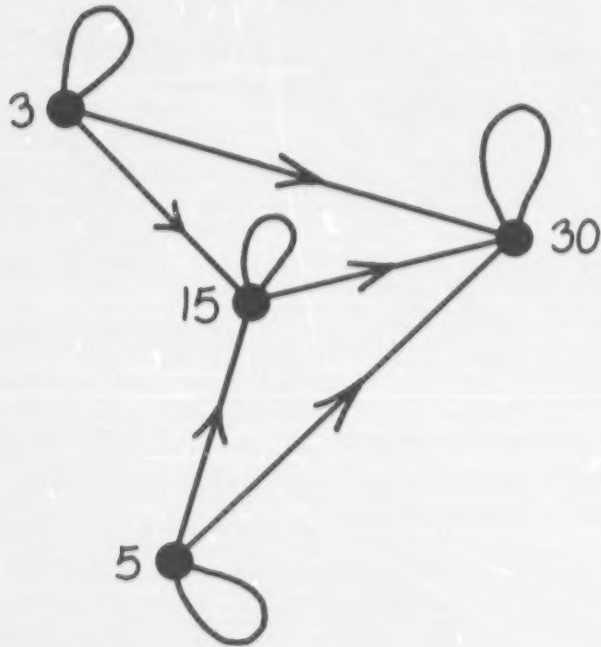
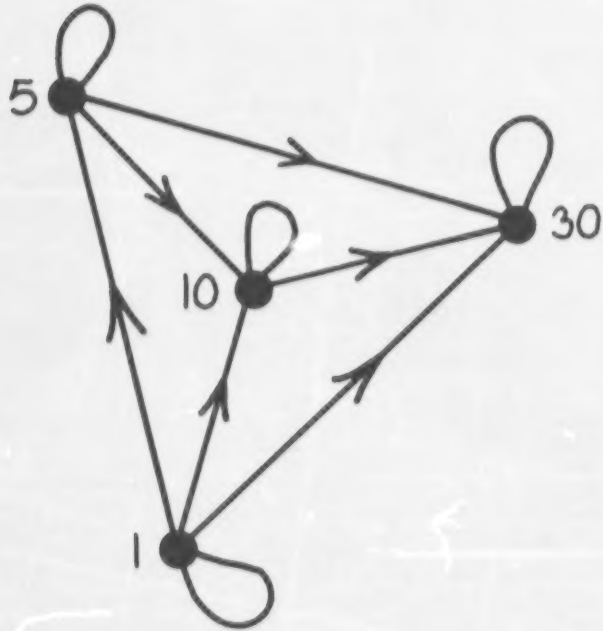
Draw on the board:



T: Copy this picture, label the dots and draw the missing blue arrows and loops.

Encourage the students to find several solutions for this problem. Let several students put their answers on the board.

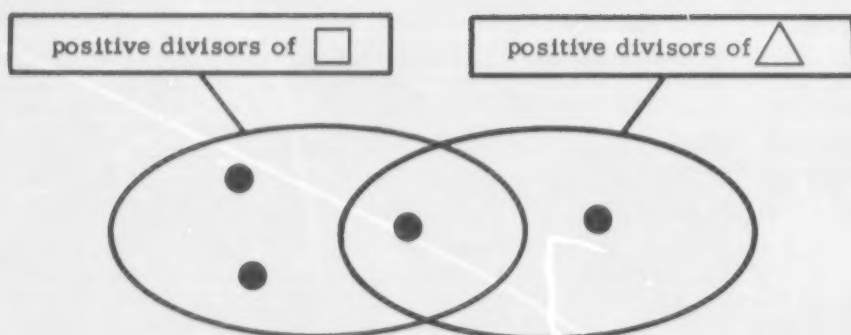
Following are two possible solutions:



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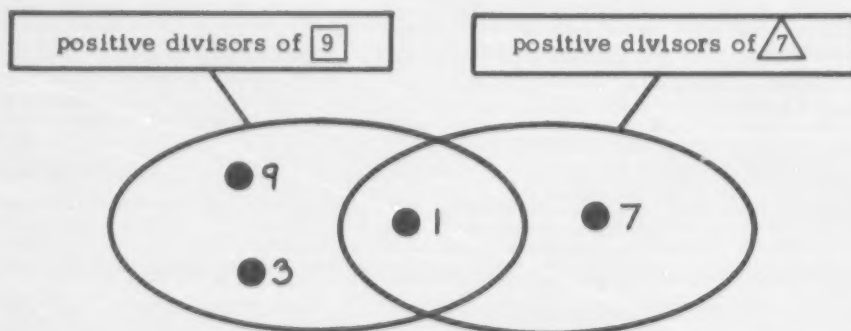
Exercise 4:

Draw on the board:



T: The blue string contains exactly three numbers and the red string exactly two. Which number could be in the little square and which number in the little triangle? Copy the picture and try to complete it. Label the dots and put a number in each frame so that the picture is correct. Many answers are possible.

After a while, let several students take turns to complete the picture. For instance,



Each time a student has found a solution, write it in this chart. Then erase the numbers in the picture and ask another student to complete the picture.

$\square$	$\triangle$
9	7

After a while, your chart should be filled with many pairs of numbers. For instance,

$\square$	$\triangle$
9	7
4	5
25	2
9	5
4	13
49	11
4	3
121	7

T: What do you notice about the numbers in the triangle?

S: They are prime numbers. They have exactly two divisors.

T: What do you notice about the numbers in the square?

S: They are square numbers ... but not just any square number. They are all squares of prime numbers.

If the students aren't able to discover the answer, explain it yourself.



prehensive School Mathematics Program

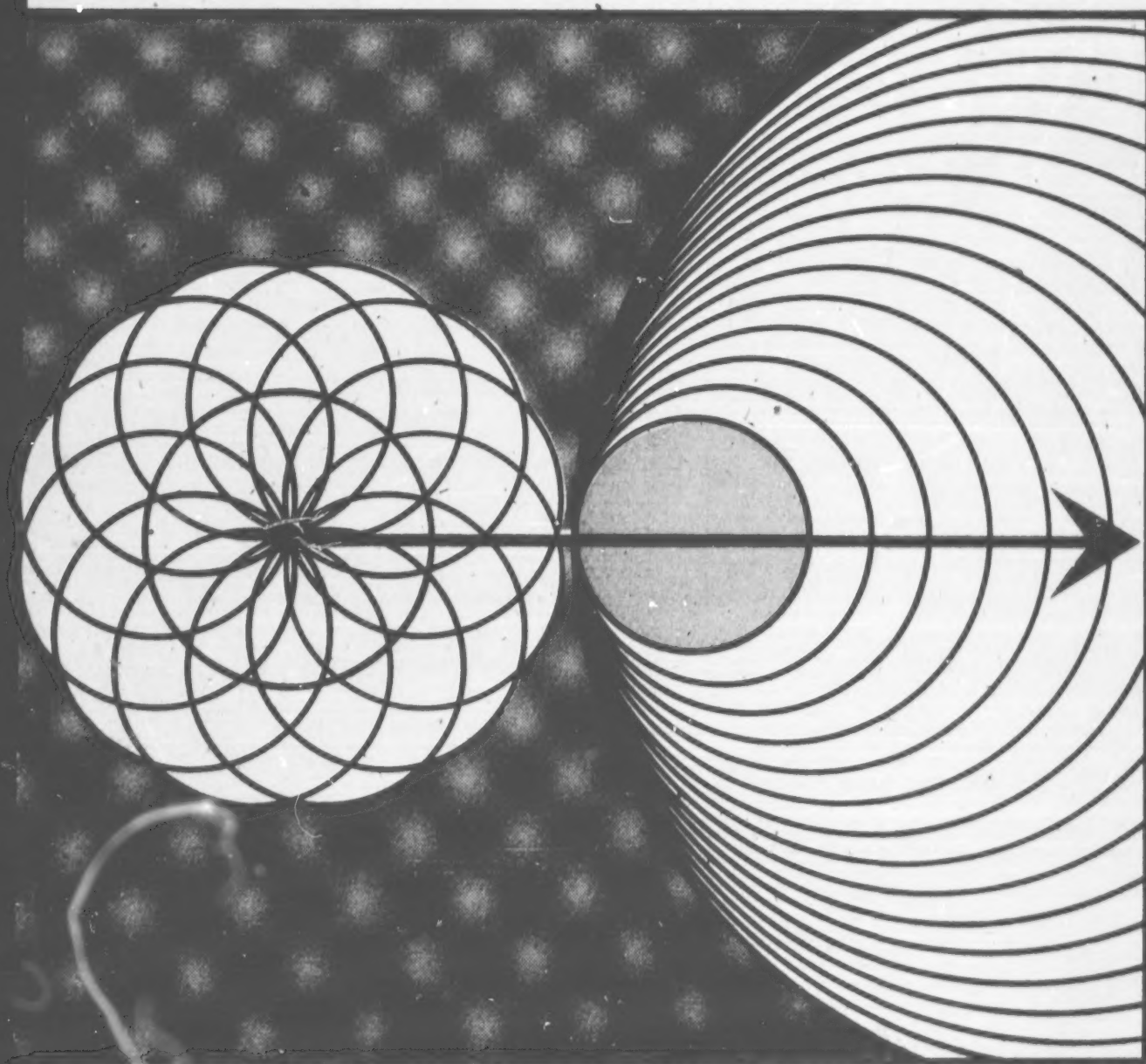
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4-19401

CSMP Mathematics for the Intermediate Grades Part I

Name _____

Worksheets



WHAT'S IN THIS BOOK?

This book contains all the worksheets you will need this semester. They are in the following order:

1. All the "N" worksheets: N01 through N49
2. All the "P" worksheets: P01 through P08
3. All the "G" worksheets: G01 through G07
4. All the "L" worksheets: L01 through L14

HOW DO YOU REMOVE PAGES?

- Step 1:** Flip through until you find the page you want.
The number of each worksheet is in the top right corner of the page.
- Step 2:** Fold and crease the page along the perforation line.
- Step 3:** With your left hand index finger and thumb, press book firmly down left of the perforation line and along center fold, starting at the top of the book.
- Step 4:** With your right hand holding top right corner of page, tear page along the perforation line. Follow the tear with your left hand holding down center fold of book.

Name _____

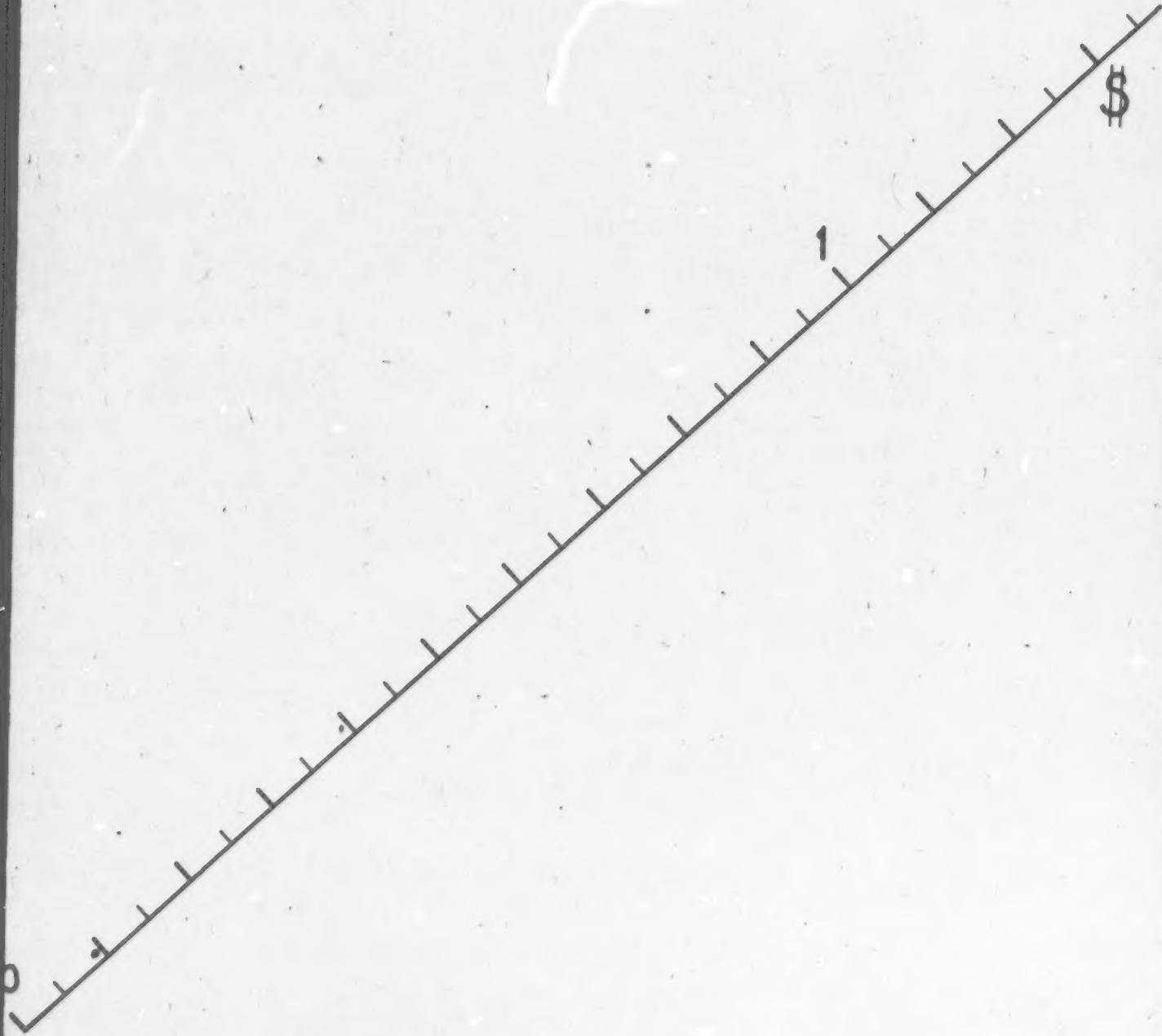
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~~873~~

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Name _____

N02



Name _____

N03

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Find all the numbers that can be put on the ones' board with two positive and two negative checkers. Write them on both sides of this worksheet.

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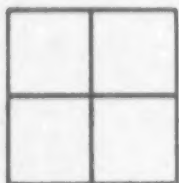
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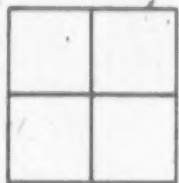
Name _____

N04

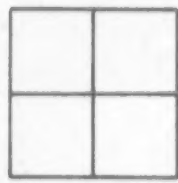
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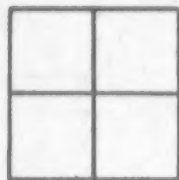
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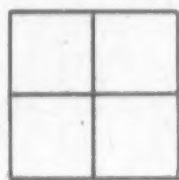
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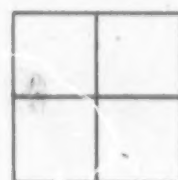
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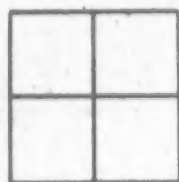
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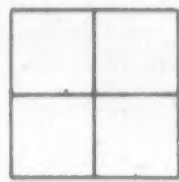
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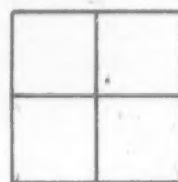
=



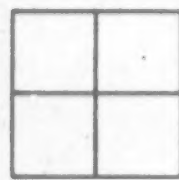
=



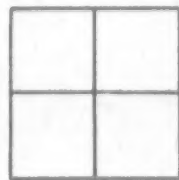
=



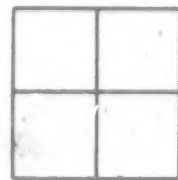
=



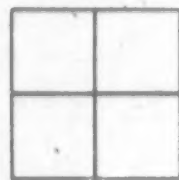
=



=



=



=



=

ame _____

N05******

Write these numbers with three checkers (positive or negative).

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = 17$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = 69$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = 37$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = 66$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = 71$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = -36$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = 73$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \hat{87}$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = -24$$

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \hat{99}$$

Name _____

N06******

In each row of this chart, write the number of items of each type that you could buy and spend \$ 1,000 altogether.

Item A: \$8	Item B: \$15	Item C: \$24	Item D: \$105

Name _____

N07

*

In each row of this chart, write the number of items of each type that you could buy and spend \$100 altogether. One solution is given for you.

Item A: \$5	Item B: \$8	Item C: \$12	Item D: \$25
6	1	1	2

Name _____

N08******

In each row of this chart, write the number of items of each type that you could buy and spend \$ 300 altogether.

Item A: \$6	Item B: \$10	Item C: \$15	Item D: \$25

Name _____

N09*******

In each row of this chart, write the number of items of each type that you could buy and spend \$ 1,000 altogether.

Item A: \$18	Item B: \$24	Item C: \$15	Item D: \$110

Name _____

N10

*

Complete:

$10 \times 45 = \underline{\hspace{2cm}}$

$2 \times 45 = \underline{\hspace{2cm}}$

$12 \times 45 = \underline{\hspace{2cm}}$

$20 \times 45 = \underline{\hspace{2cm}}$

$22 \times 45 = \underline{\hspace{2cm}}$

$18 \times 45 = \underline{\hspace{2cm}}$

$40 \times 45 = \underline{\hspace{2cm}}$

$39 \times 45 = \underline{\hspace{2cm}}$

$80 \times 45 = \underline{\hspace{2cm}}$

$100 \times 45 = \underline{\hspace{2cm}}$

Name _____

N11

**

Complete:

$10 \times 54 = \underline{\hspace{2cm}}$

$3 \times 54 = \underline{\hspace{2cm}}$

$13 \times 54 = \underline{\hspace{2cm}}$

$30 \times 54 = \underline{\hspace{2cm}}$

$43 \times 54 = \underline{\hspace{2cm}}$

$60 \times 54 = \underline{\hspace{2cm}}$

$59 \times 54 = \underline{\hspace{2cm}}$

$100 \times 54 = \underline{\hspace{2cm}}$

$103 \times 54 = \underline{\hspace{2cm}}$

$130 \times 54 = \underline{\hspace{2cm}}$

$133 \times 54 = \underline{\hspace{2cm}}$

Name _____

N12*******

$$3 \times 46 = 138 \quad . \quad 10 \times 46 = 460$$

Complete:

$$1) \quad 460 + 138 = \boxed{} \times 46$$

$$2) \quad 460 - 138 = \boxed{} \times 46$$

$$3) \quad 460 + 138 + 138 = \boxed{} \times 46$$

$$4) \quad 460 + 460 + 460 = \boxed{} \times 46$$

$$5) \quad 3 \times 460 = \boxed{} \times 46$$

$$6) \quad 460 + 46 = \boxed{} \times 46$$

$$7) \quad 460 - 46 = \boxed{} \times 46$$

$$8) \quad 4,600 = \boxed{} \times 46$$

$$9) \quad 4,600 + 138 = \boxed{} \times 46$$

$$10) \quad 4,600 + 460 = \boxed{} \times 46$$

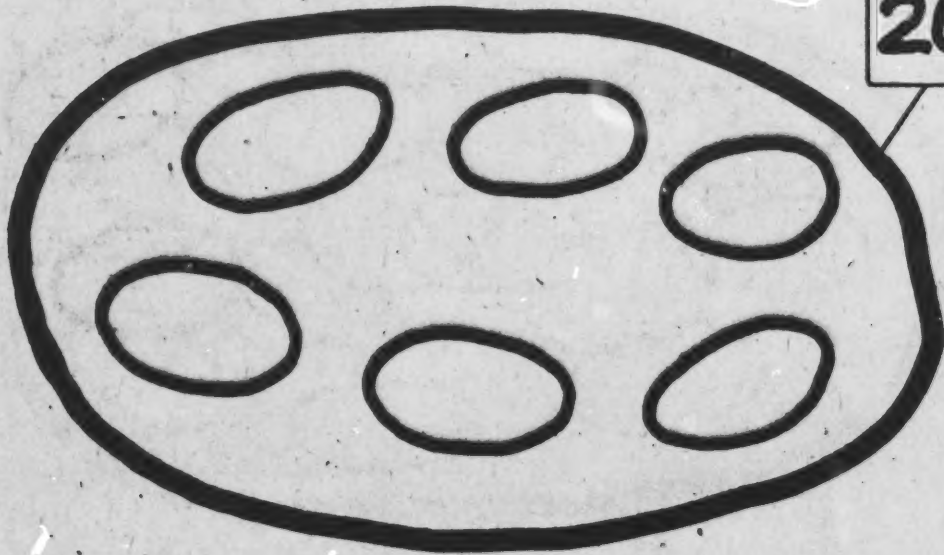
$$11) \quad 4,600 + 460 + 138 = \boxed{} \times 46$$

Name _____

N13

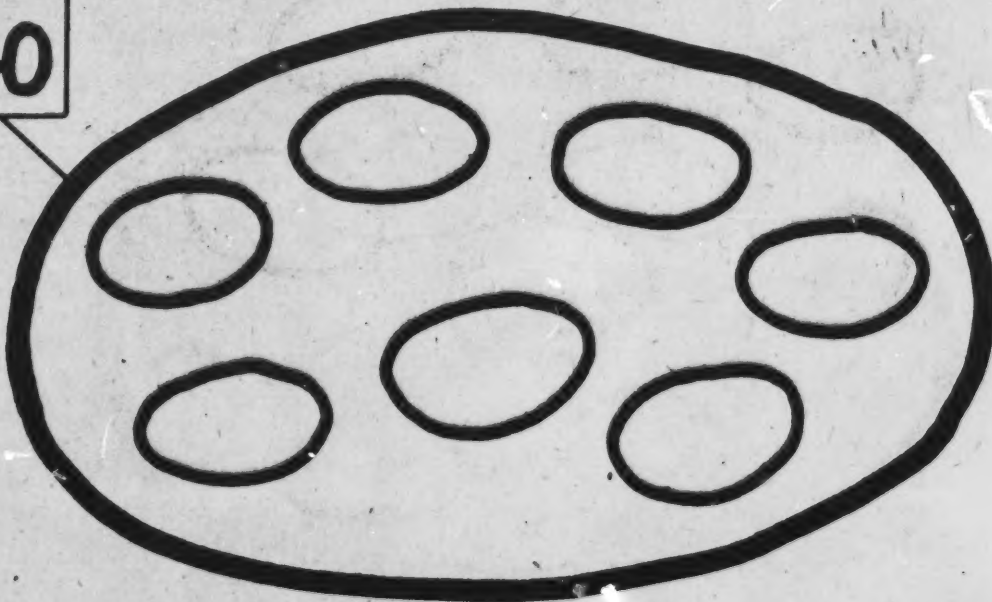
*

20



Calculation: _____

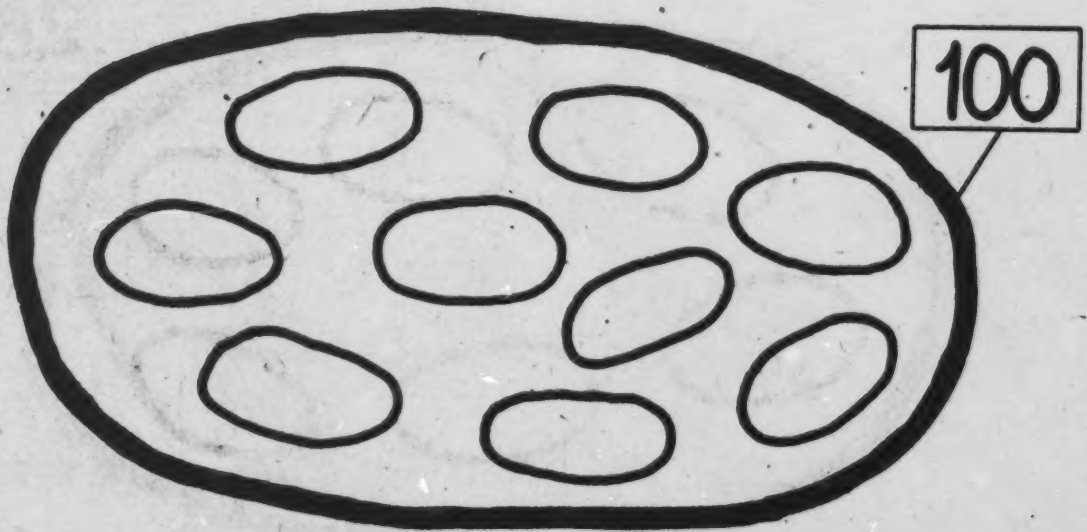
40



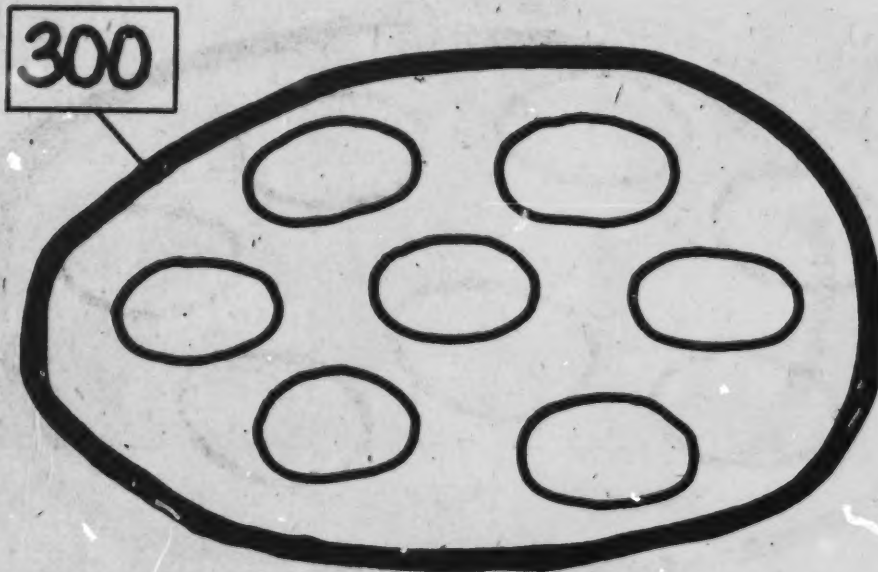
Calculation: _____ 391

Name _____

N14



Calculation: _____

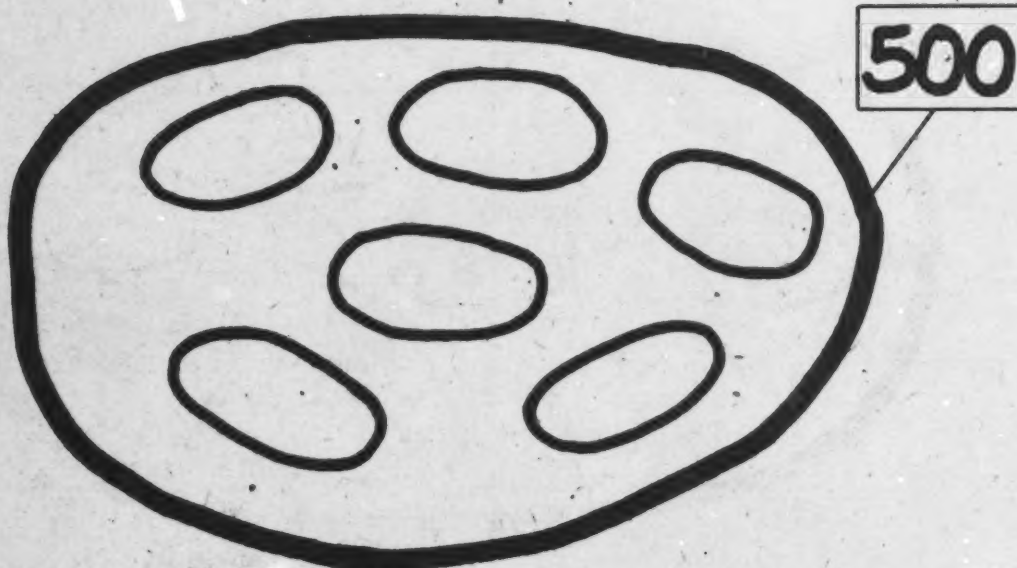


Calculation: _____

392

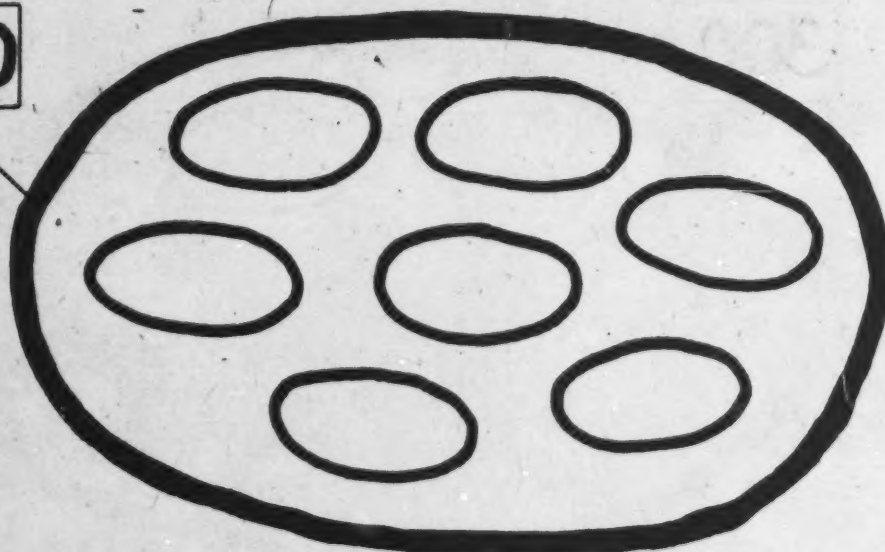
Name _____

N15



Calculation: _____

1,000



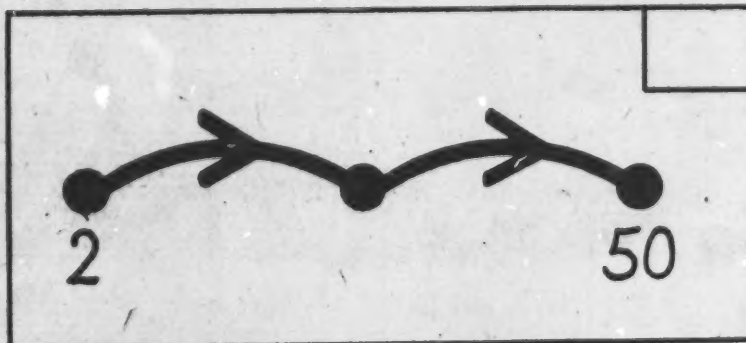
Calculation: _____

Name _____

N16

*

For each arrow picture, fill in the box in the upper right-hand corner with one of these tags. Then label the dots.

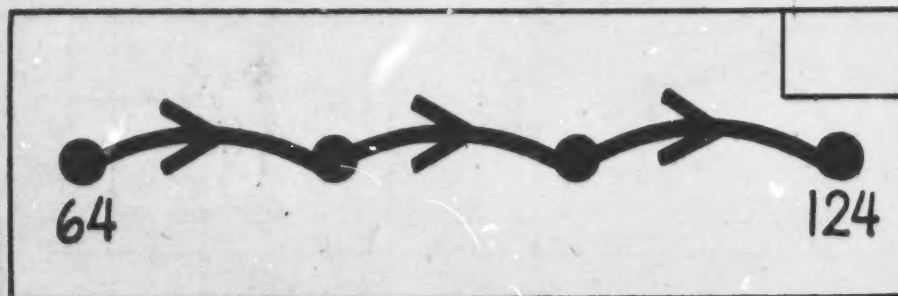
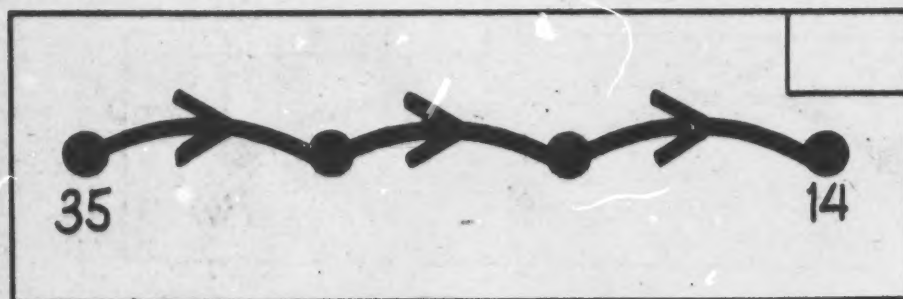
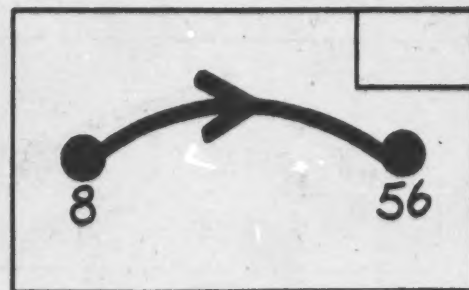
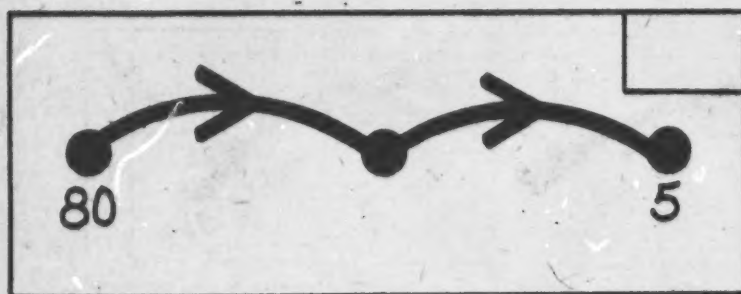
+5 **$\times 2$** **-4** **$\times 5$** 

Name _____

N17

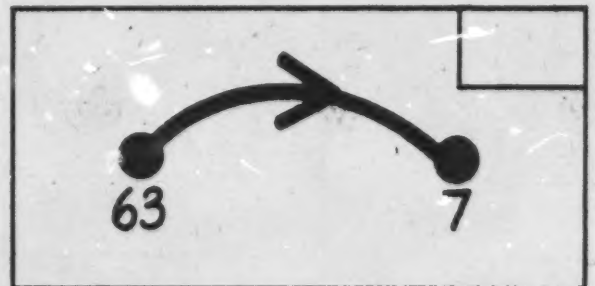
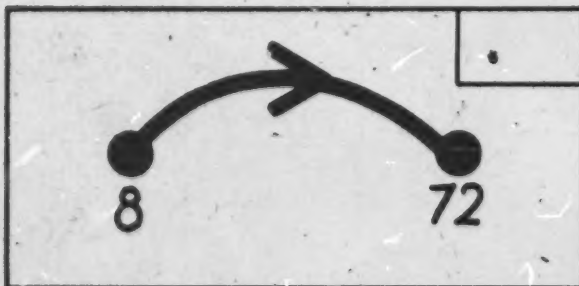
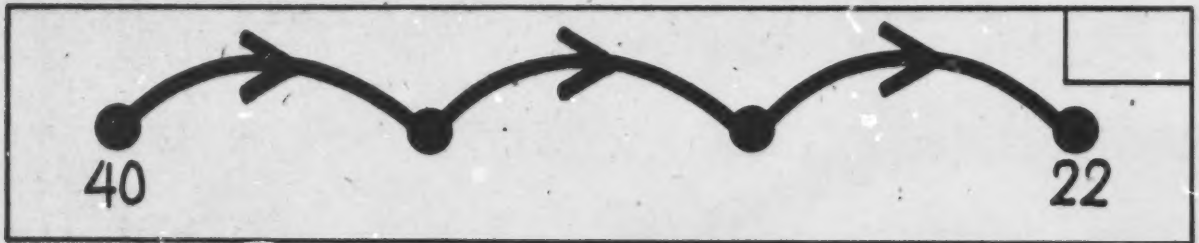
**

For each arrow picture, fill in the box in the upper right-hand corner with one of these tags. Then label the dots.

 -7 $+20$ $\times 7$ $\div 4$ $\times 3$ 

N18*********Name** _____

In each of these pictures fill in the box in the upper right-hand corner with a label for the arrows. Then label the dots.



Name _____

N19

*



Match :

 $\times 3$ $\div 2$ -3 $+90$ $\times 7$ -9 $\times 10$ $\div 2$ -20 $\times 50$ $+20$ $+43$

Name _____

N 20

**



Complete the chart:

→	→
+30	
÷2	
-20	
×2	

→	→
	×2
	+20
	÷2
	×100
-15	
-25	

Name _____

N21

*



Match:

 $\times 2$ -20 -5 -40 $+30$ $+48$ $\times 3$ $+10$ -4 $+52$ -6 $+50$

Name _____

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N 22

**



Complete the chart:

→	→
+20	
-20	
÷2	
	+50
	+40
	+80

→	→
-30	
-40	
	÷2
400	×10

Name _____

N23



Complete the chart:

$\rightarrow$	$\rightarrow$
$\times 2$	
$\times 20$	
	$\div 2$
	$\times 2$
$\times 3$	
$\times 30$	

$\rightarrow$	$\rightarrow$
-30	
-20	
	$\times 10$
	-25

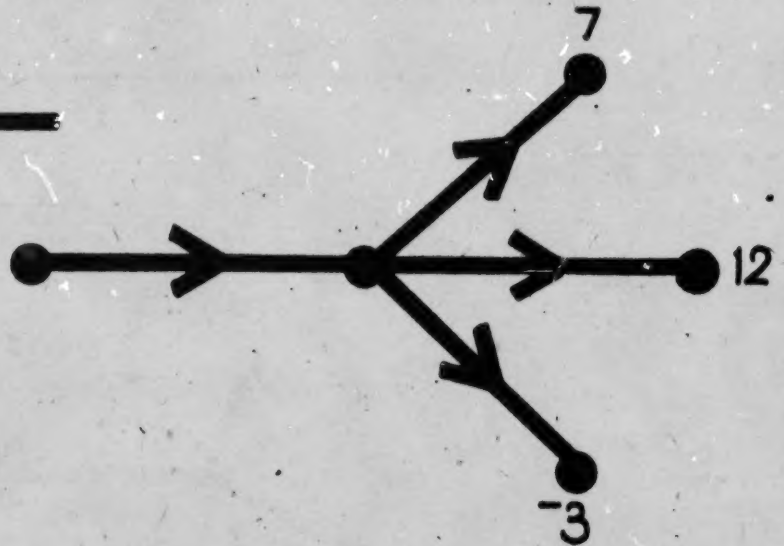
Name _____

N24

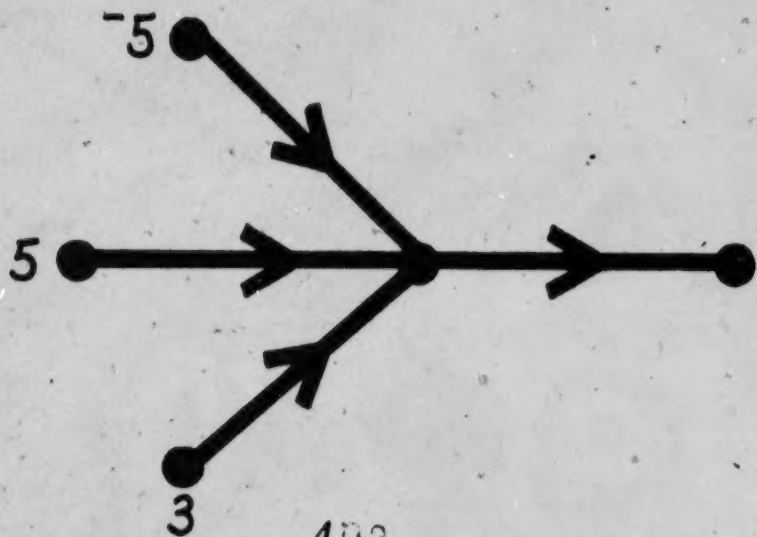
*

In both pictures, label the dots. Many answers are possible. Then draw all the missing arrows.

I am larger than you



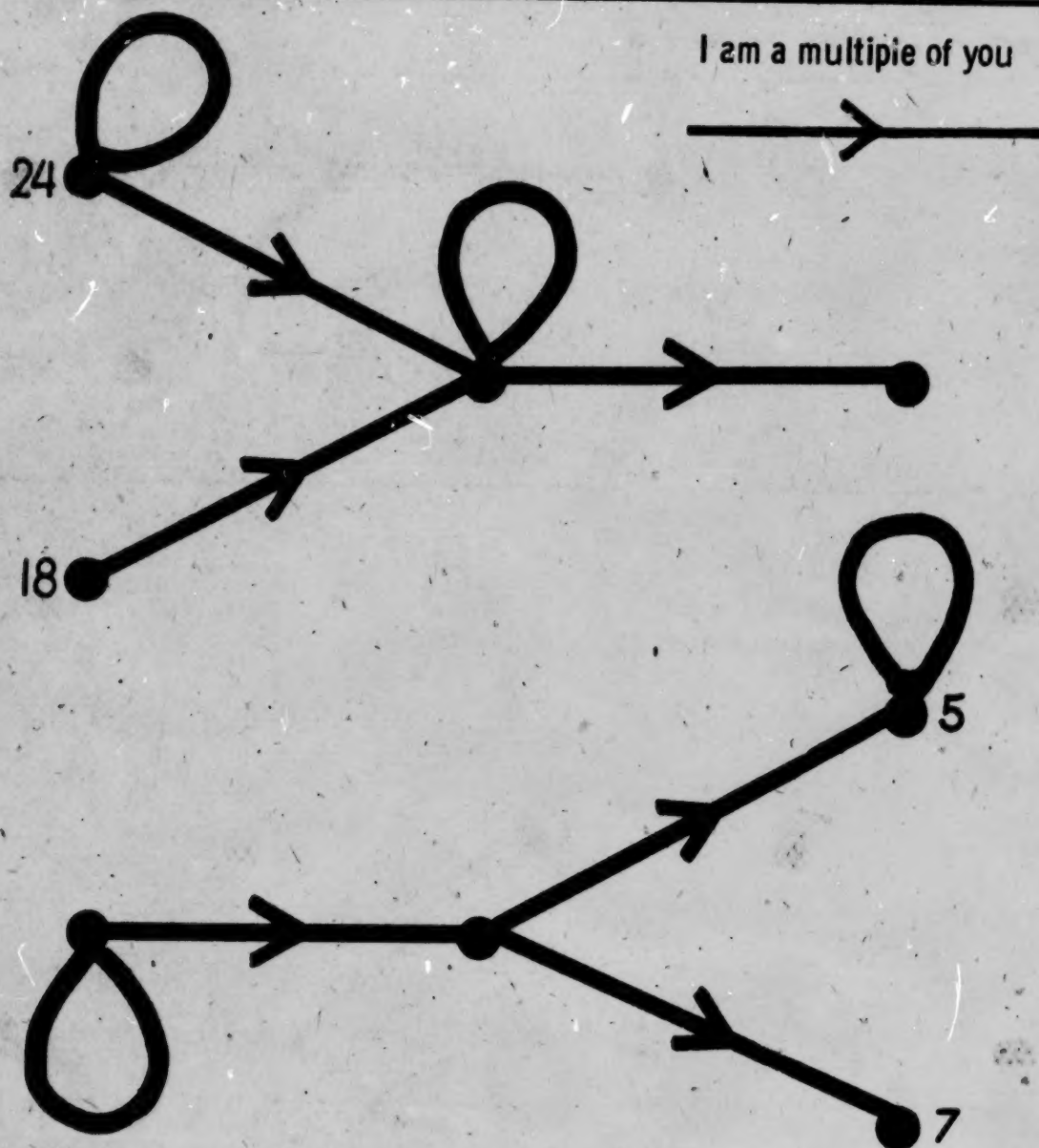
I am at least 10 smaller than you



Name _____

N 25******

Label the dots. Many answers are possible. Then draw all the missing arrows.



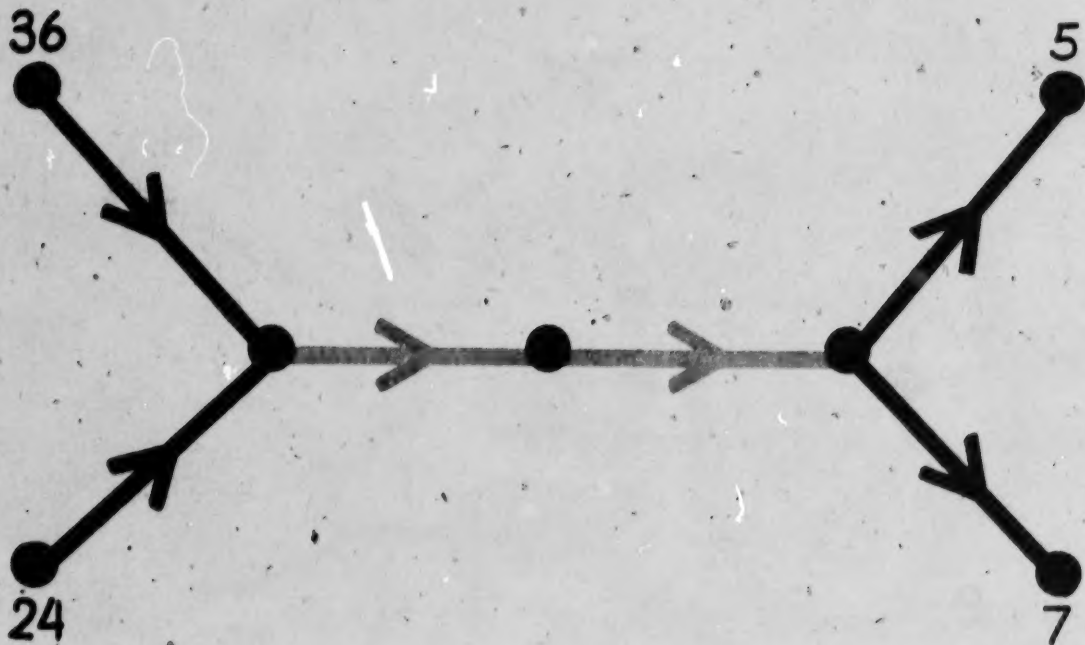
Name _____

N 26******

Label the dots. Many answers are possible. Then draw all the missing arrows. (It's not necessary to draw the red loops as it would make the picture too crowded.)

I am a multiple of you

I am at least 20 smaller than you



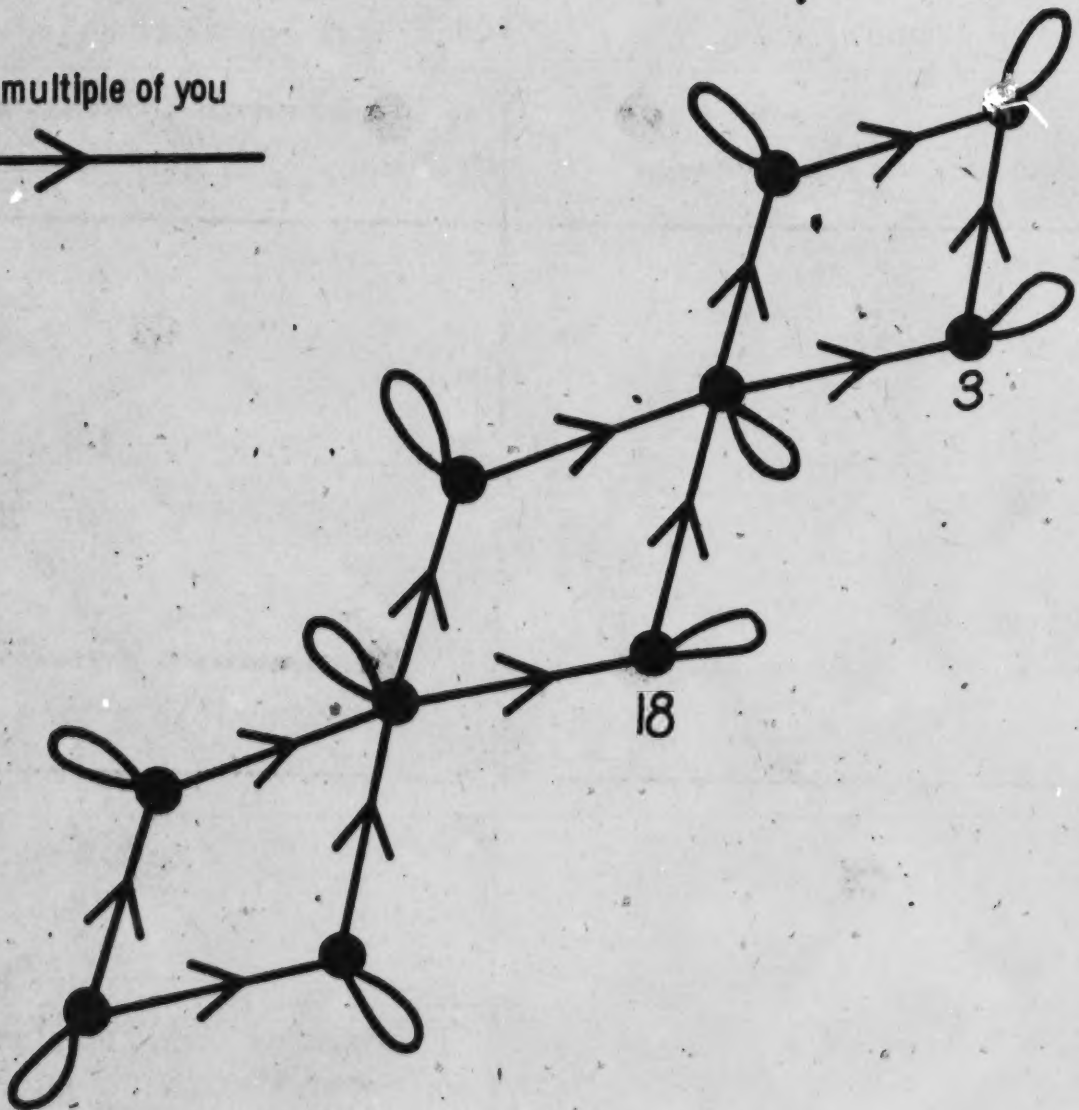
N 27

Name _____

Label the dots. Many answers are possible. Trace with your finger all the missing arrows but don't draw them.

How many missing arrows did you trace? _____

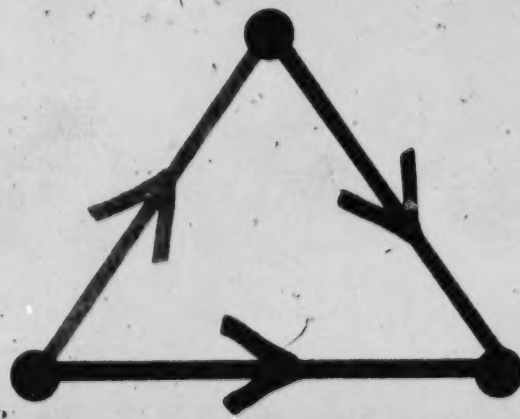
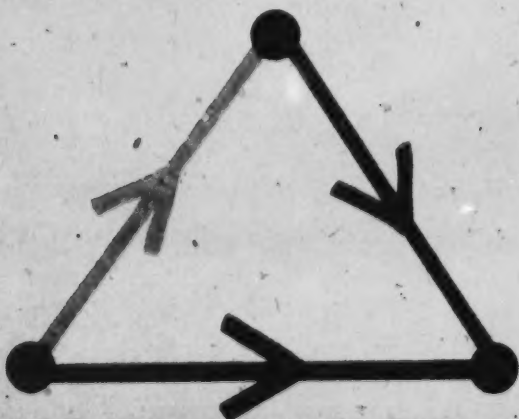
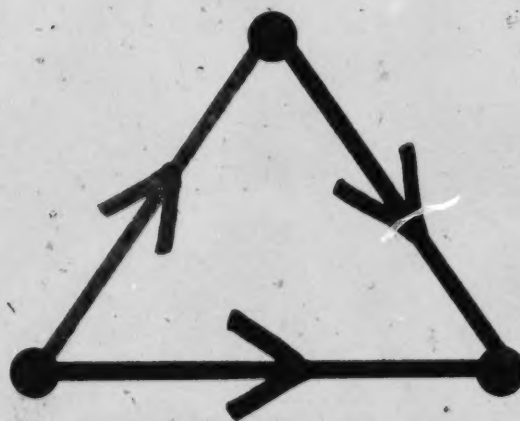
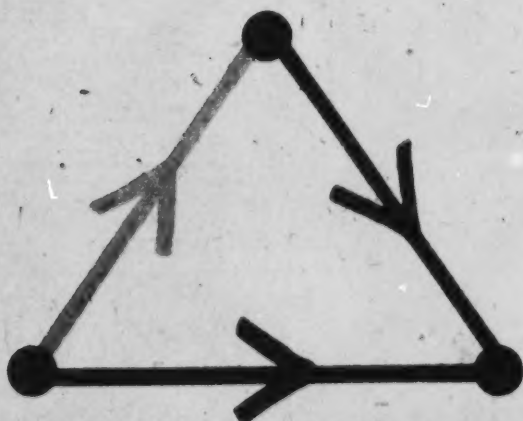
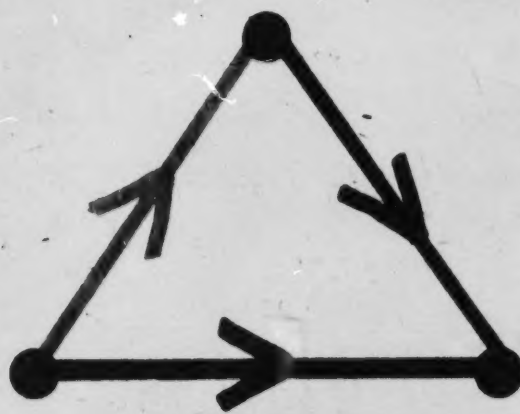
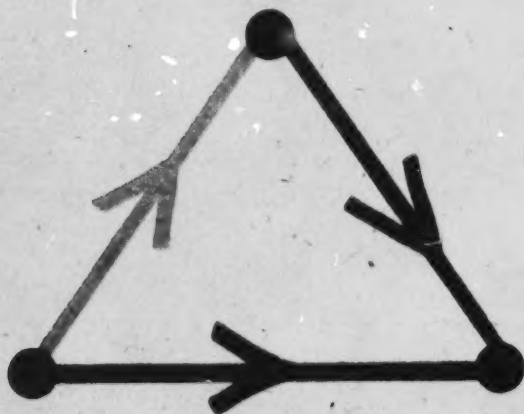
I am a multiple of you



Name _____

N28

*

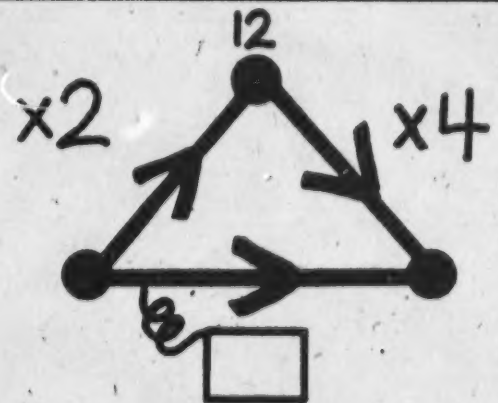
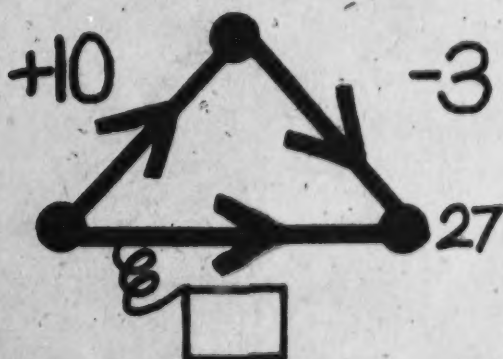
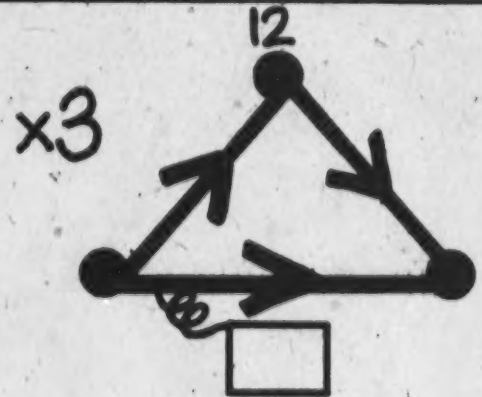
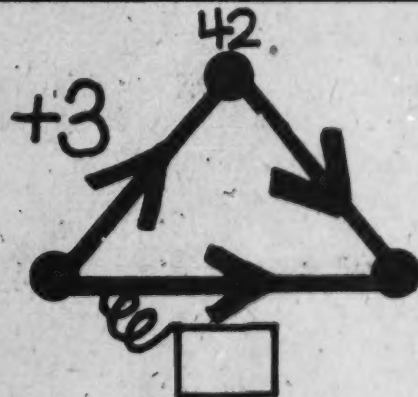
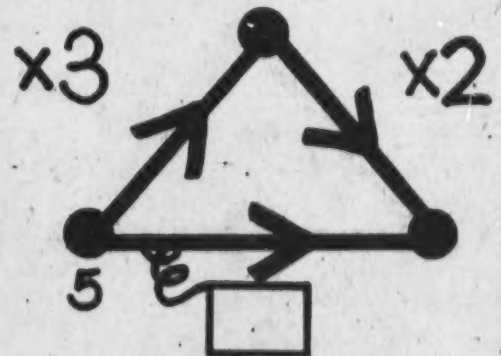
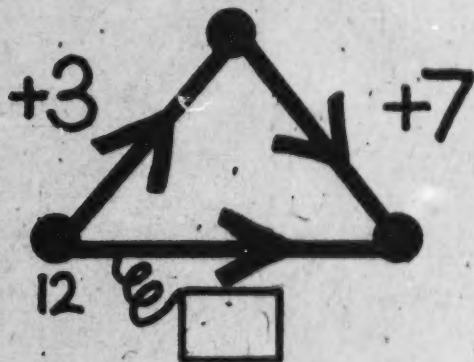


Name _____

N 29

*

Label the dots and complete the tag for each green arrow with one of the following tags:

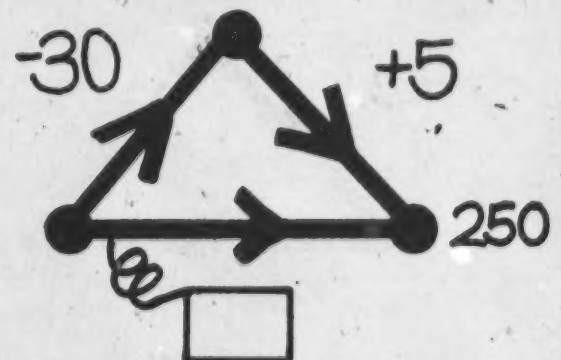
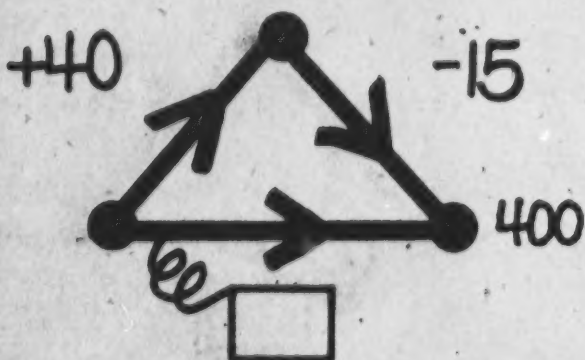
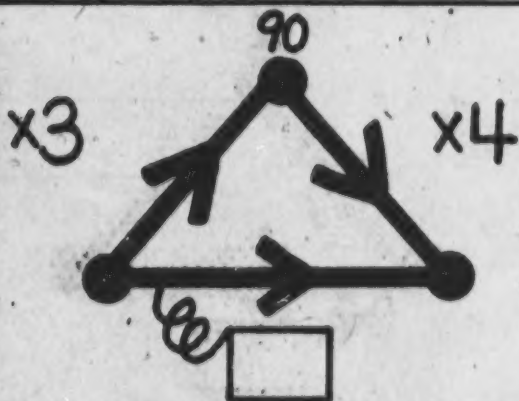
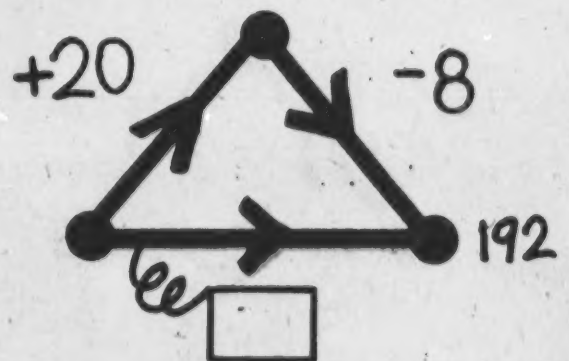
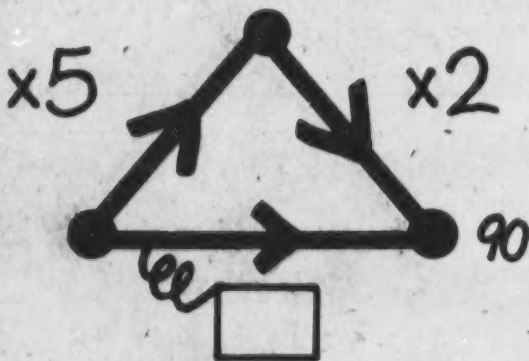
x6**x9****+10****+7****x8****+6**

Name _____

N 30

**

Label the dots and complete the tag for each green arrow with one of the following tags:

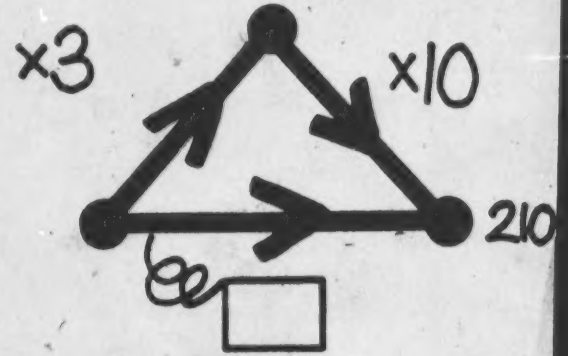
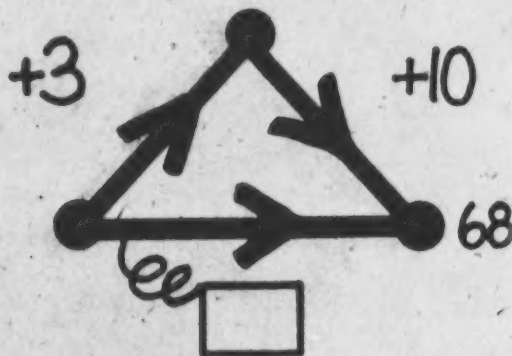
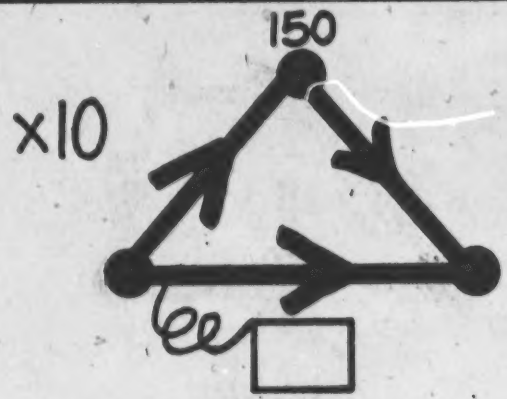
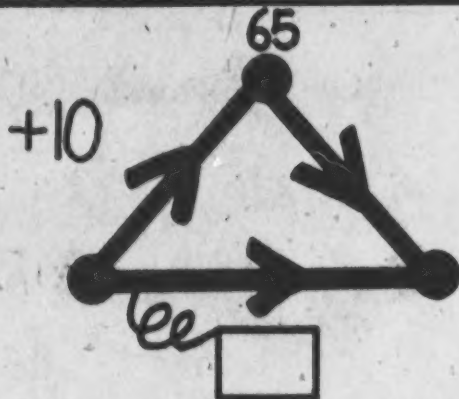
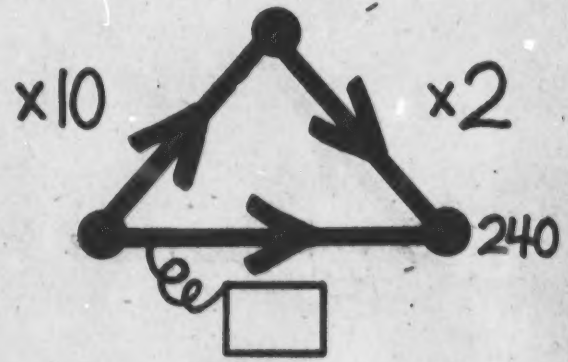
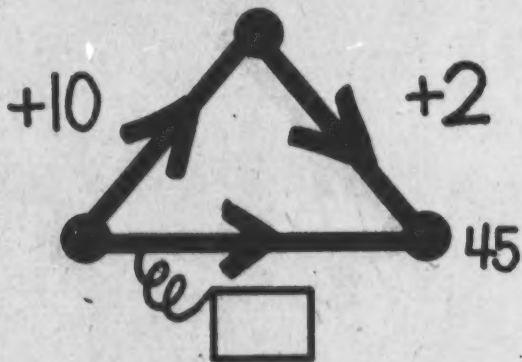
 $\times 12$ $\times 25$ -25 $\times 10$ $+25$ $+12$ 

Name _____

N 31

*

Label the dots and complete the tag for each green arrow with one of the following tags:

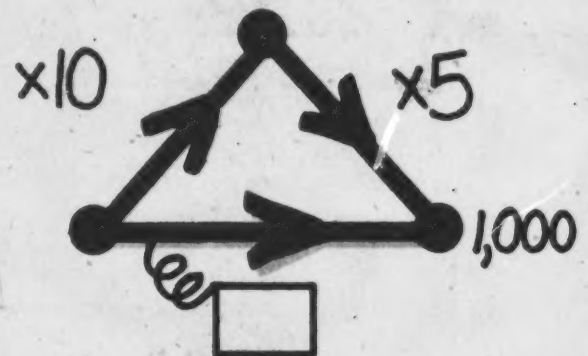
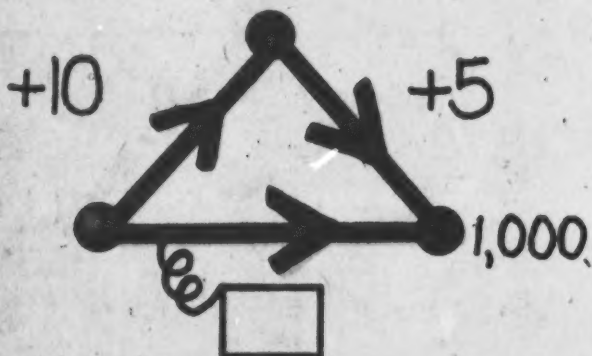
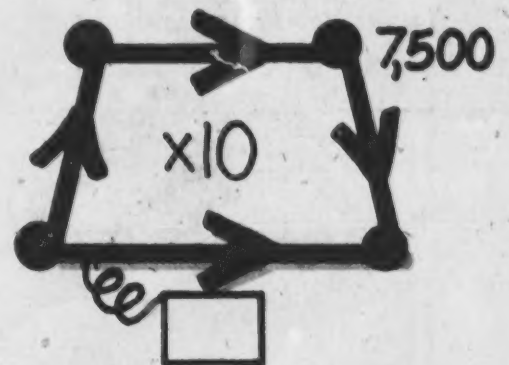
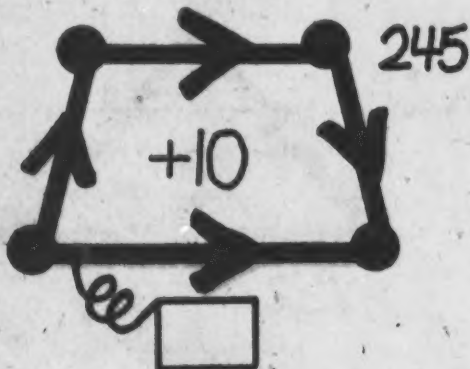
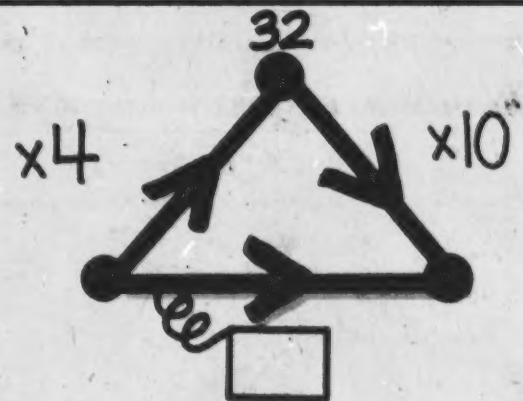
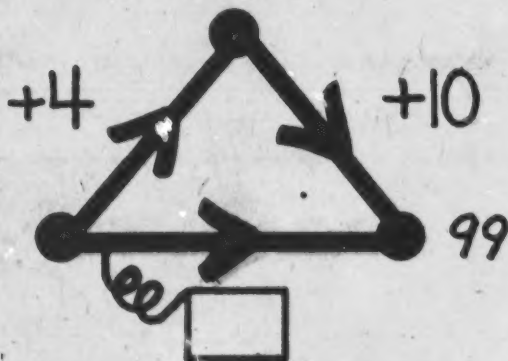
 $\times 100$ $\times 20$ $+13$ $+20$ $+12$ $\times 30$ 

Name _____

N32

**

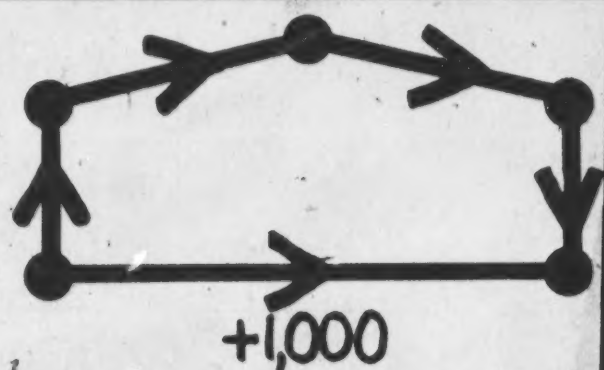
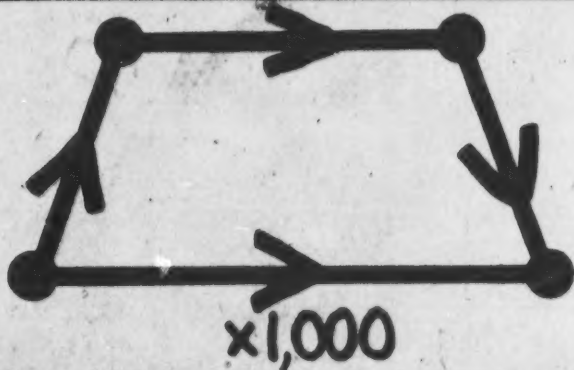
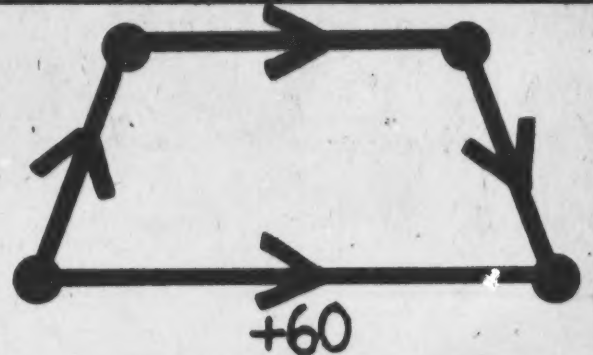
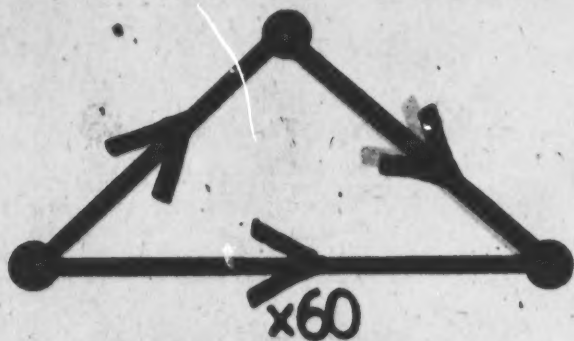
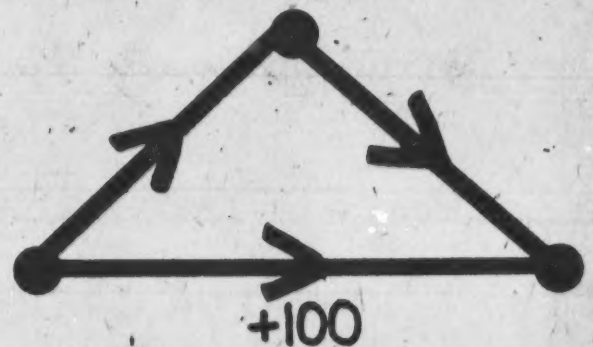
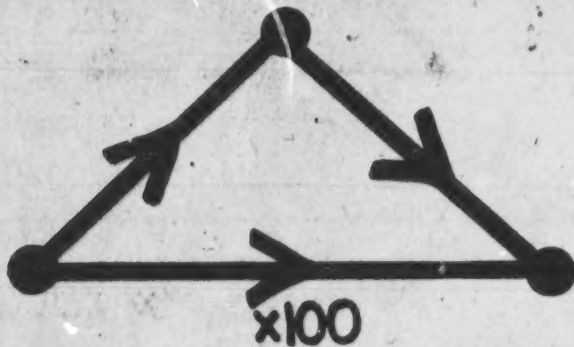
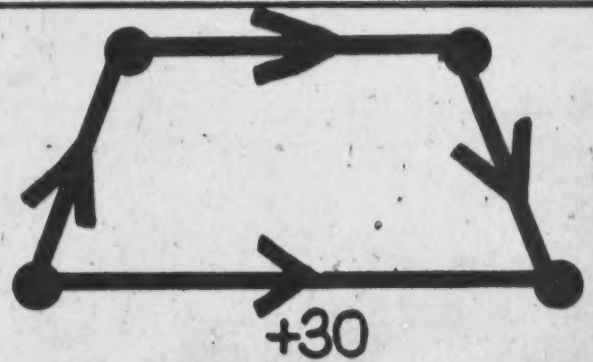
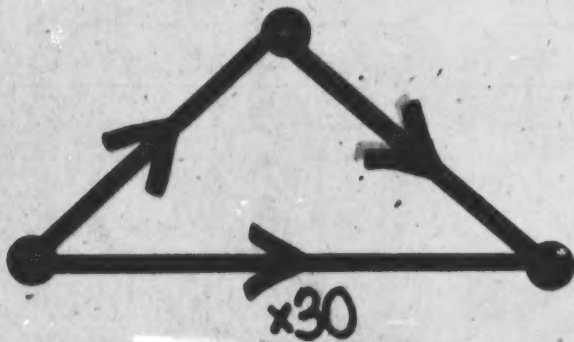
Label the dots and complete the tags for each green arrow with one of the following tags:

 $\times 40$ $+15$ $\times 1,000$ $+14$ $\times 50$ $+30$ 

Name _____

N33

Label the blue arrows, the green arrows and the dots. Many solutions are possible.

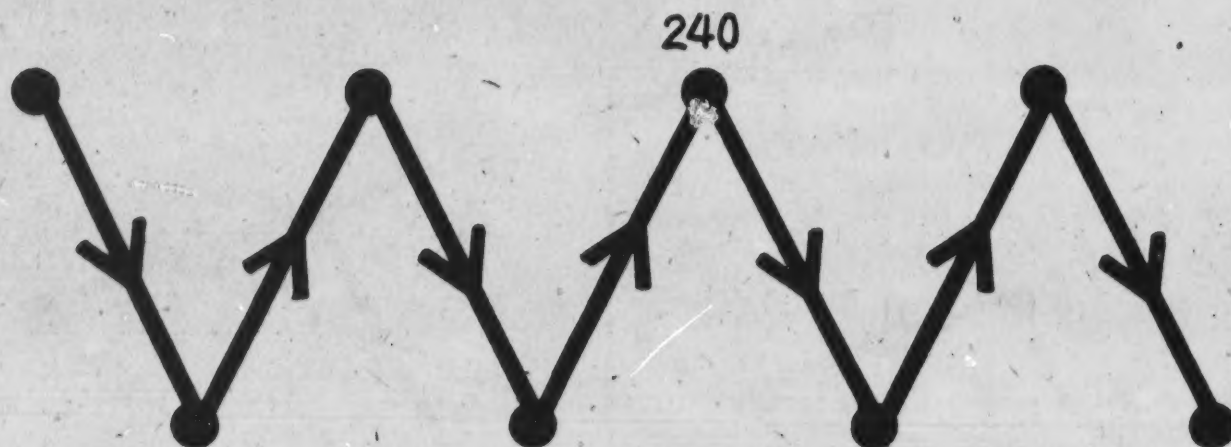
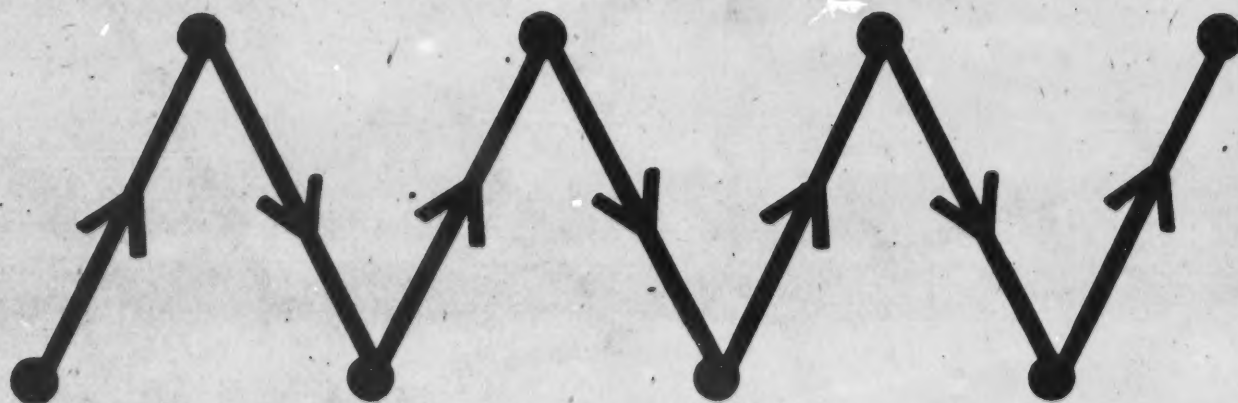


Name _____

N34

*

Label all the dots. Use green to draw as many x20 arrows as you can.

 $\times 2$ $\times 10$ 

400

412

Name _____

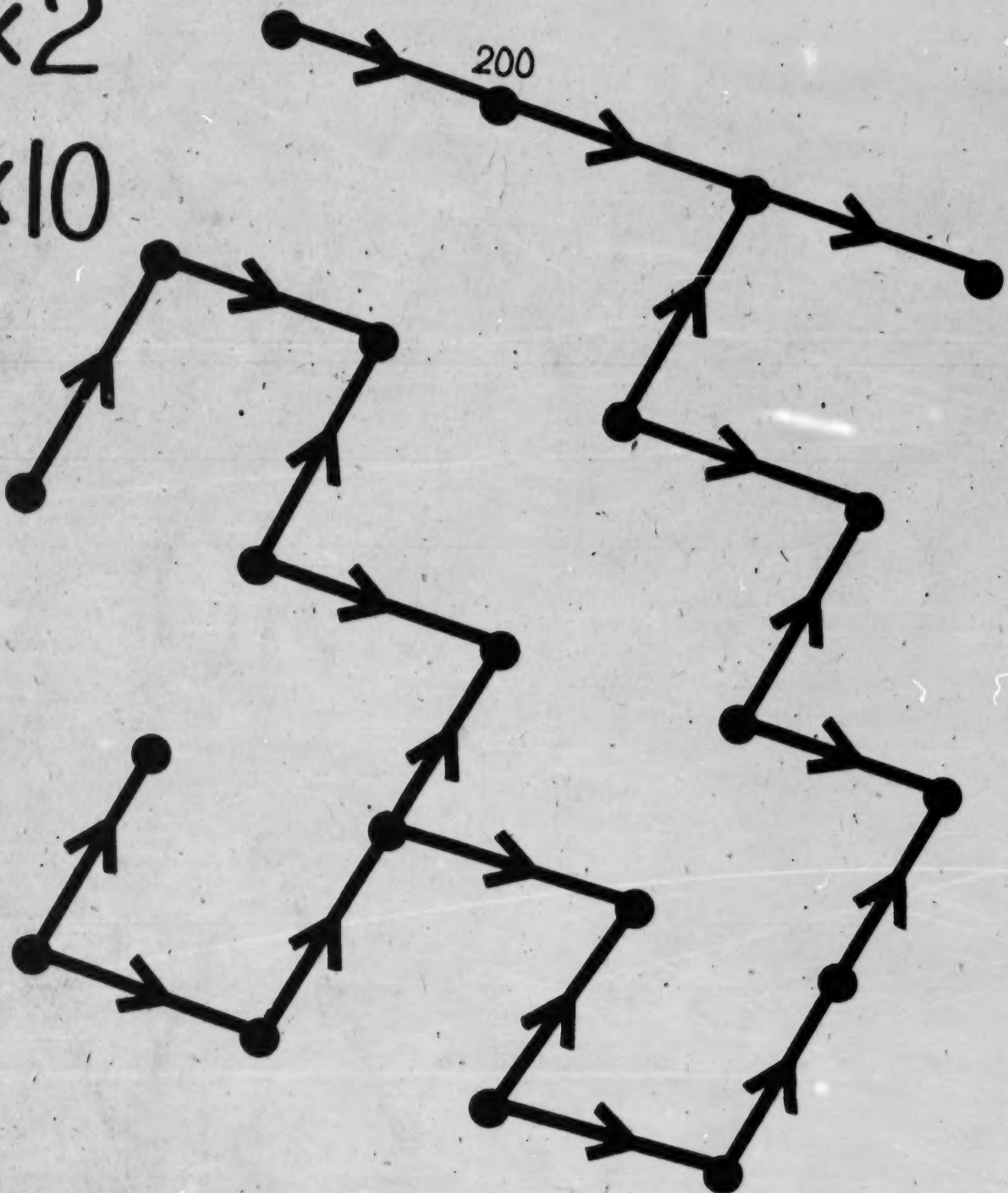
N 35

**

Label all the dots. Draw the missing red arrows and the missing blue arrows. Use green to draw as many x20 arrows as you can.

x2

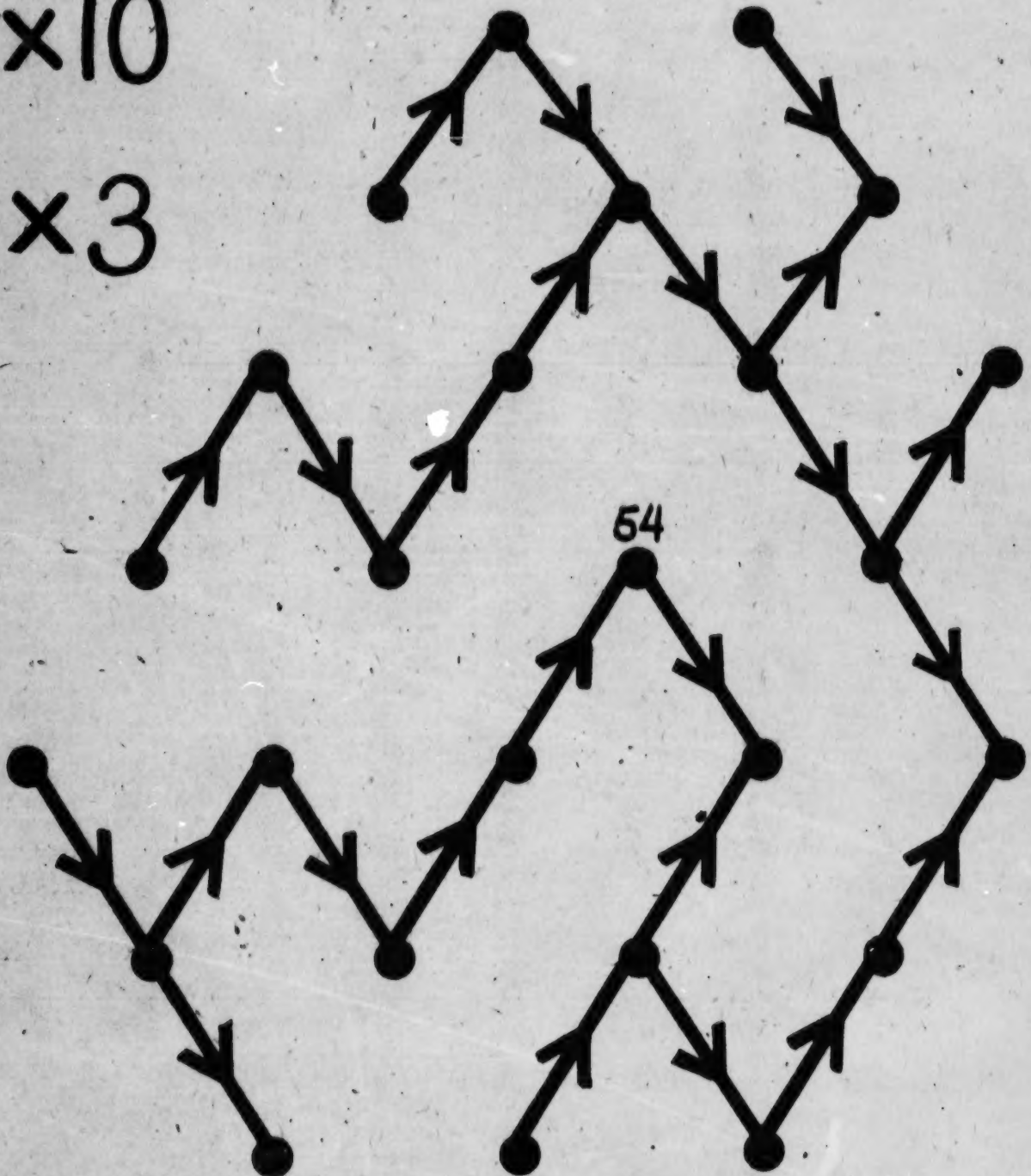
x10



N36

Name _____

Label all the dots. Draw the missing red arrows and the missing blue arrows. Use green to draw as many x30 arrows as you can.

 $\times 10$ $\times 3$ 

Name _____

N37

*

Complete:

$$8 * 2 =$$

$$2 * 8 =$$

$$10 * 6 =$$

$$6 * 10 =$$

$$50 * 4 =$$

$$4 * 50 =$$

$$100 * 25 =$$

$$25 * 100 =$$

$$\square * 6 = 20$$

$$6 * \square = 20$$

$$6 * \square = 10$$

$$\square * 7 = 27$$

$$7 * \square = 27$$

$$\square * 6 = 6$$

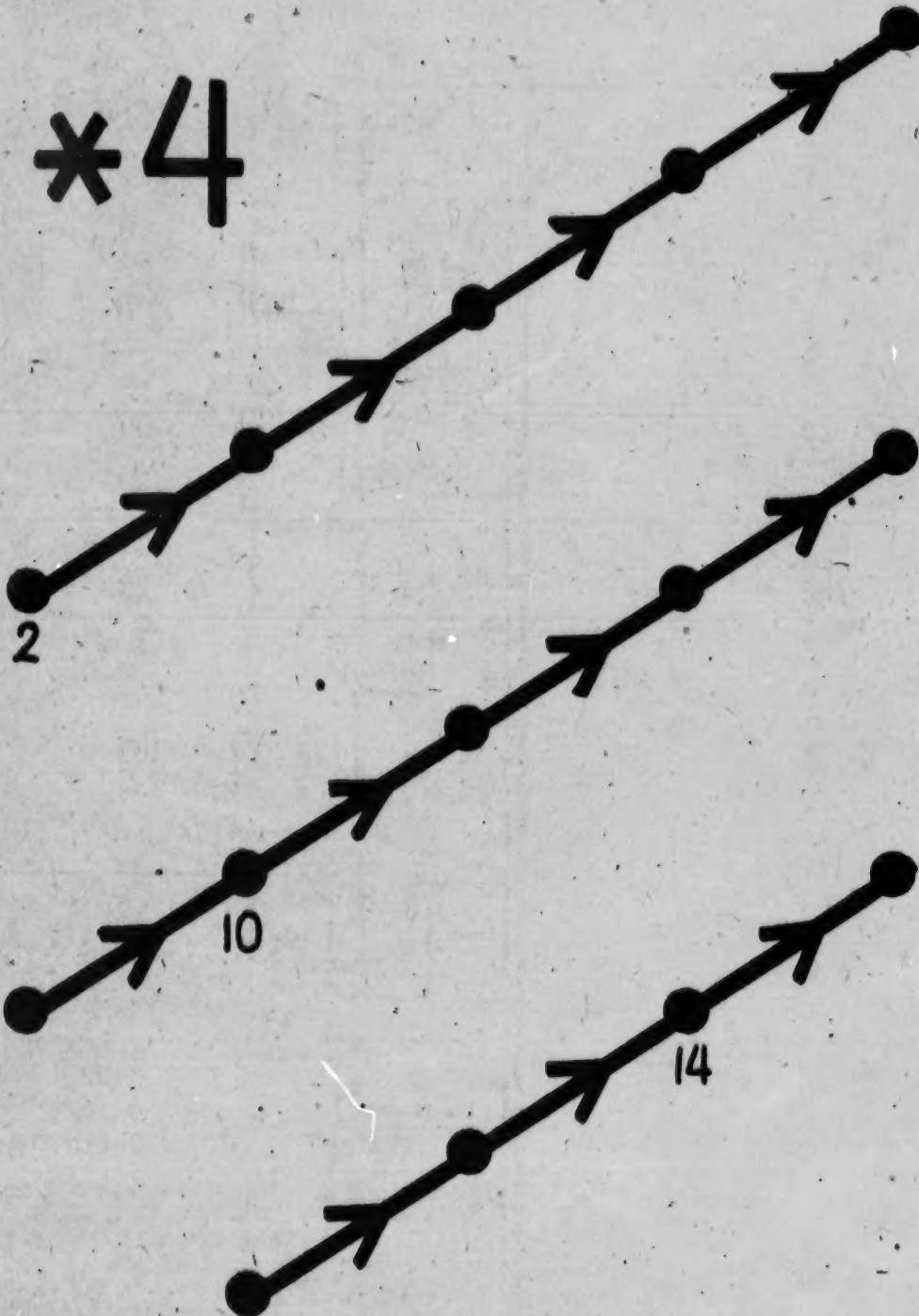
$$6 * \square = 6$$

Name _____

N38

Label the dots:

***4**



4:6

Name _____

N 39 *

Complete:

$10 * 40 =$

$40 * 10 =$

$100 * 20 =$

$20 * 100 =$

$8 * 8 =$

$12 * 12 =$

$30 * 30 =$

$\square * 20 = 14$

$30 * \square = 65$

$\square * 40 = 30$

$50 * \square = 102$

$\square * 10 = 105$

$7 * \square = 20$

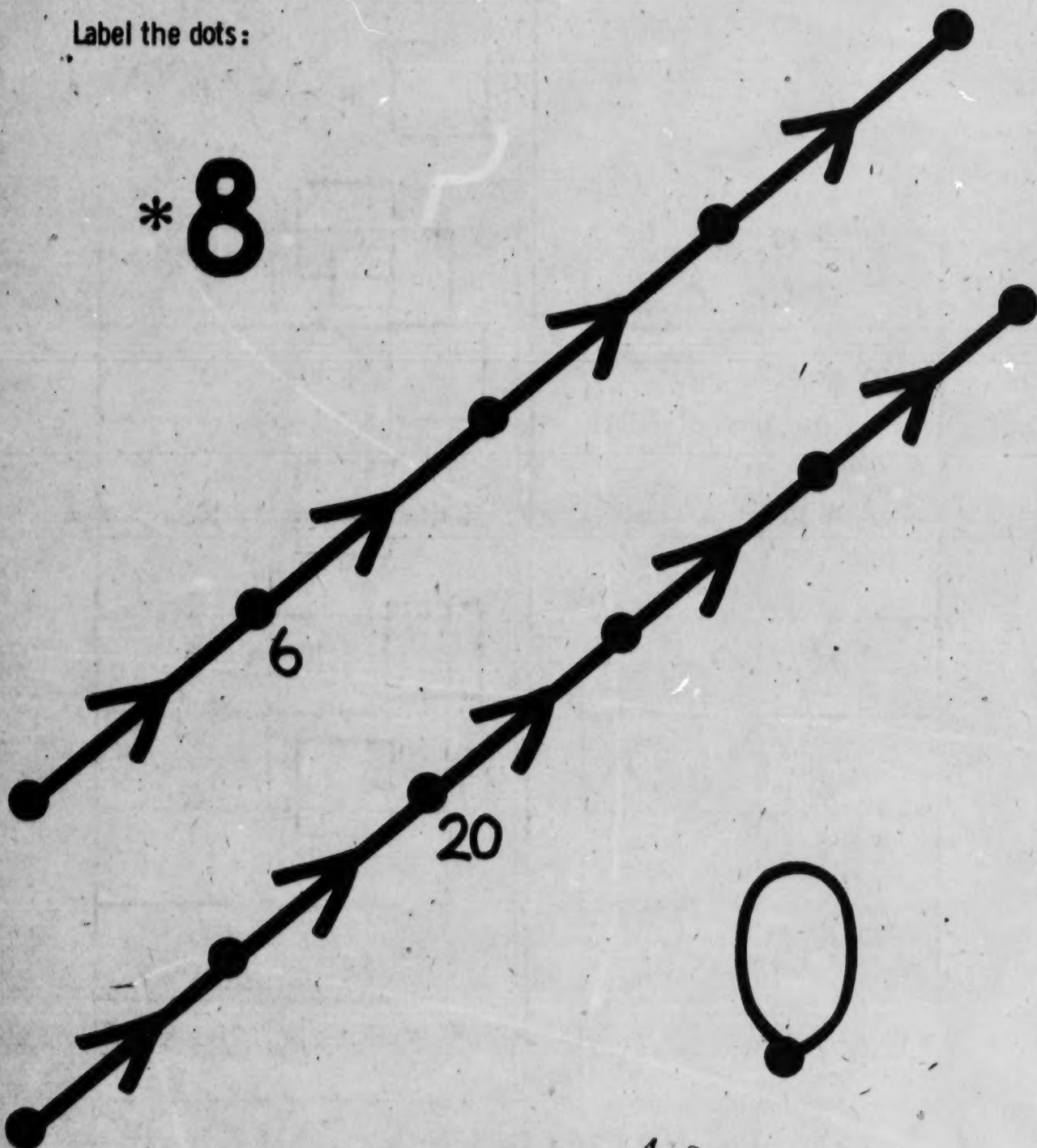
Name _____

N40

**

Label the dots:

* 8



Name _____

N 41 ***

$$\square * \triangle = 20$$

Complete the charts:

$\square$	$\triangle$
6	16
8	
	20
2	
	100
-10	
	10
10	
	-10

$\square$	$\triangle$

Name _____

N42

*

Write the 21 numbers that can be put on the ones' board with EXACTLY two positive checkers and one negative checker.

=

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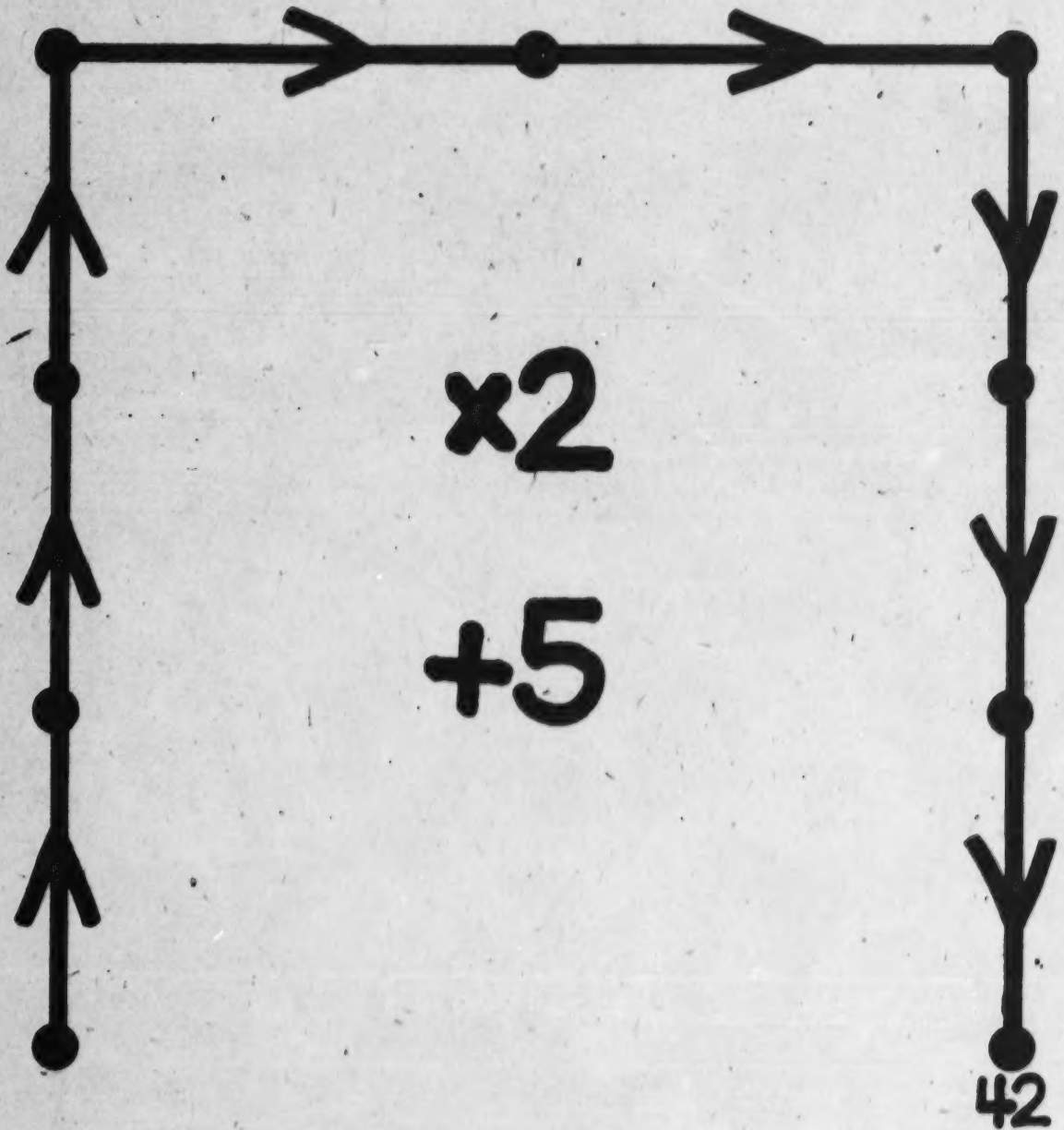
=

Name _____

N43

**

Label the dots:



Now you know that the secret number is _____.

421

Name _____

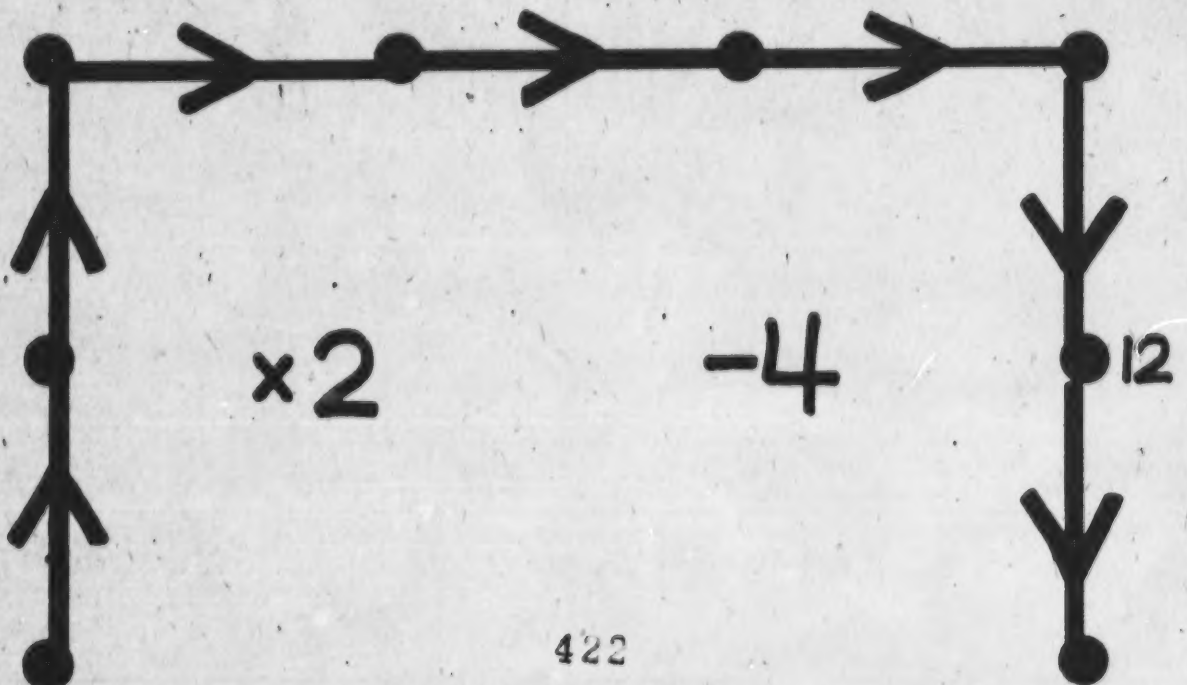
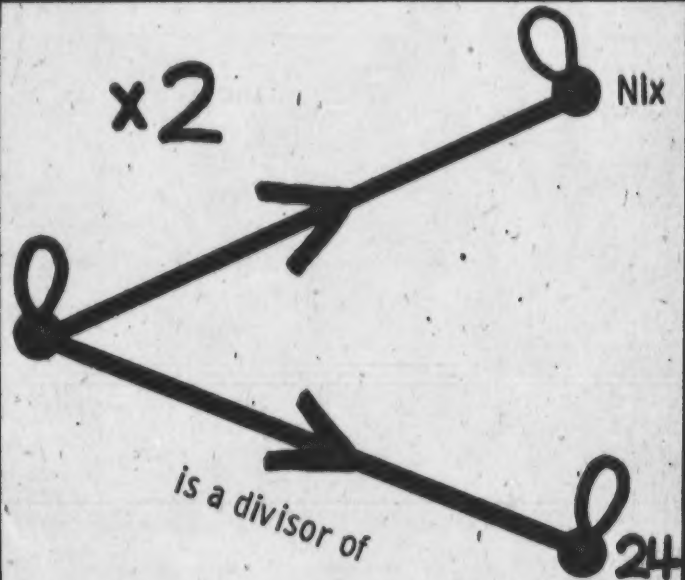
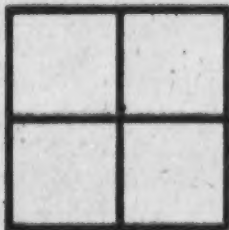
N44

*

Nix

Nix has exactly four positive divisors.

Nix is on this Minicomputer board using exactly two checkers (positive or negative). You cannot see the checkers because they are invisible.

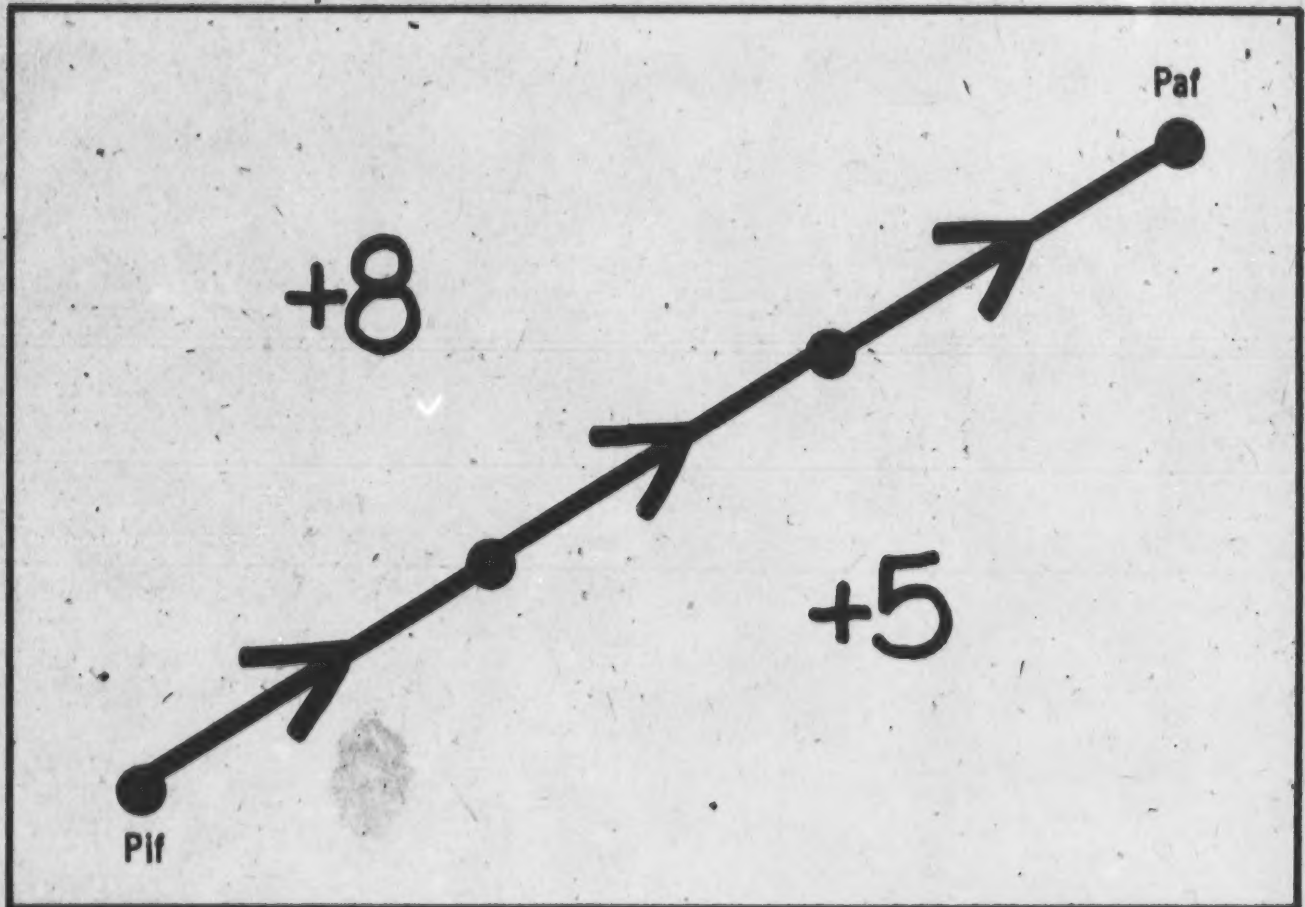


Nix is in this picture.

Name _____

N 45

Pif and Paf



Pif is a positive divisor of 72.

It is impossible for Paf to be shown on the Minicomputer using exactly 3 checkers (positive or negative).

Name _____

N46*****

Could Nack be 7? _____

Could Nack be 15? _____

Could Nack be 36? _____

Could Nack be 92? _____

Could Nack be 320? _____

Could Nack be 340? _____

Could Nack be 970? _____

Could Nack be 980? _____

Could Nack be 7, 920? _____

Could Nack be 7, 910? _____

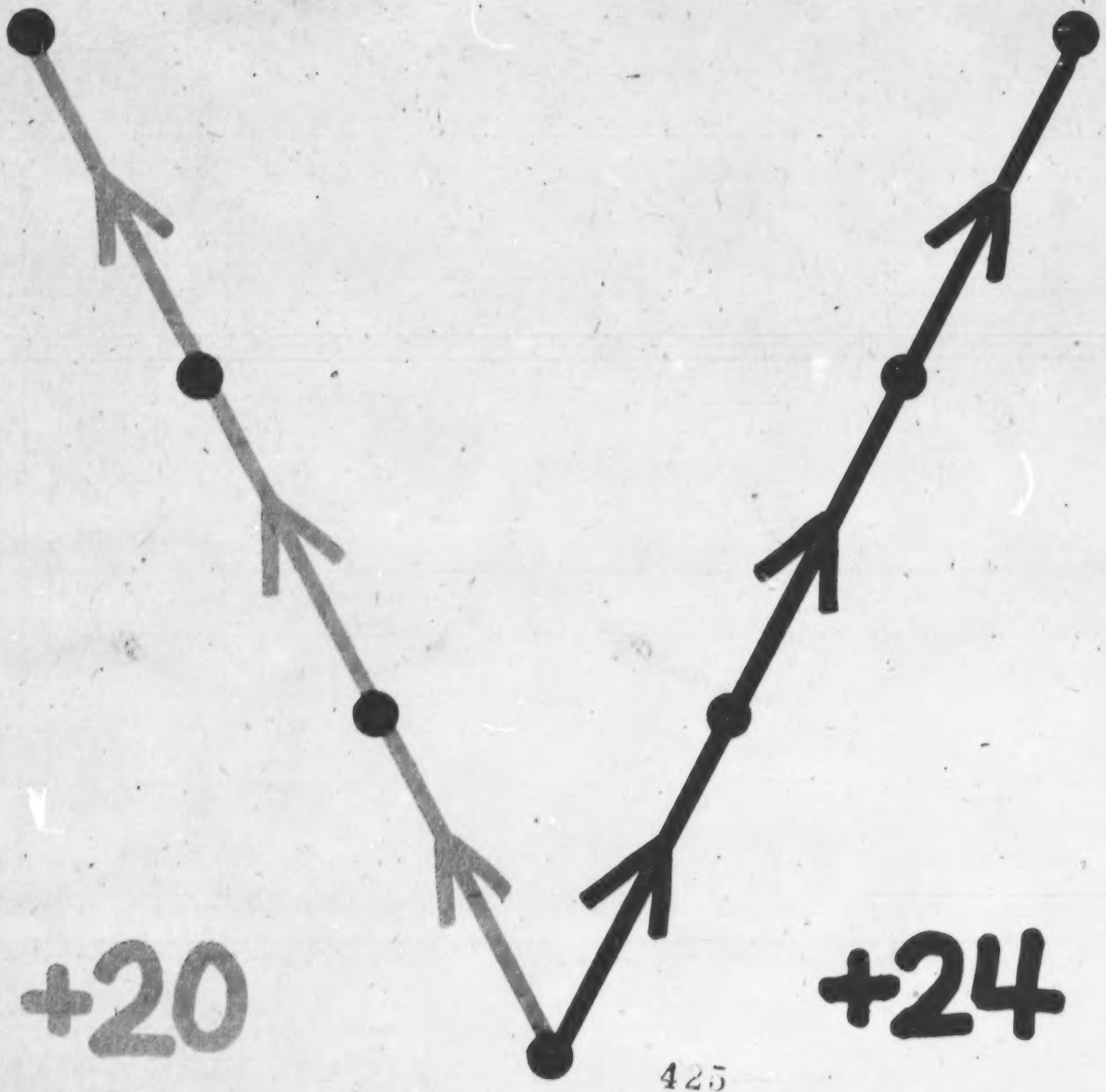
Name _____

N47

*

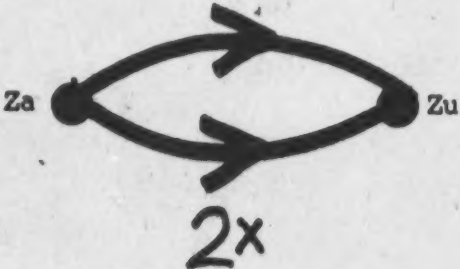
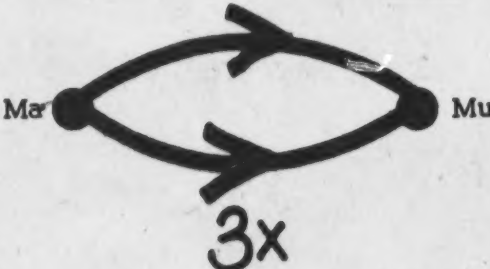
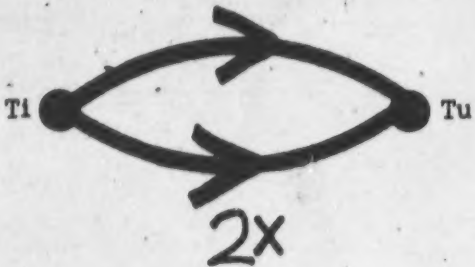
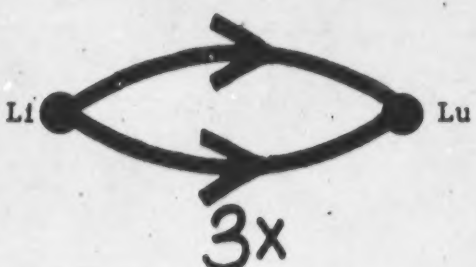
Third clue:

Nick and Nack are the largest two numbers in this picture:



Name _____

N48

+8  2x Za is _____ Zu is _____	+40  3x Ma is _____ Mu is _____
+40  2x Ti is _____ Tu is _____	+100  3x Li is _____ Lu is _____

Name _____

N 49 $+10$ $2x$

Zip

Zap

Zip is _____

Zap is _____

 $+20$ $3x$

Tip

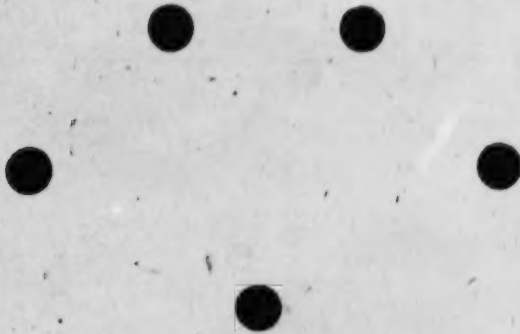
Tap

Tip is _____

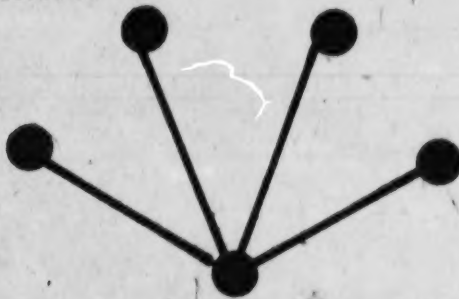
Tap is _____

EXAMPLE

Here are one red and four blue marbles.

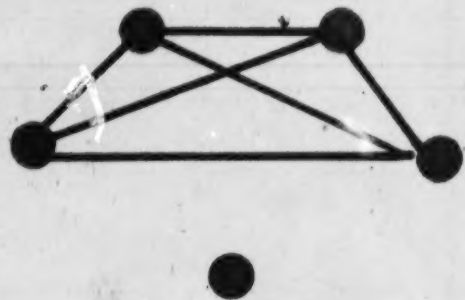


Here are all possible ways of taking two marbles of different colors.



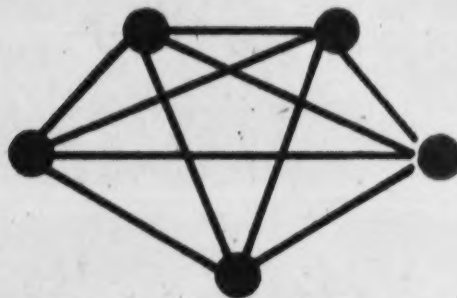
4 possible ways

Here are all possible ways of taking two marbles of the same color.



6 possible ways

Here are all possible ways of taking two marbles.



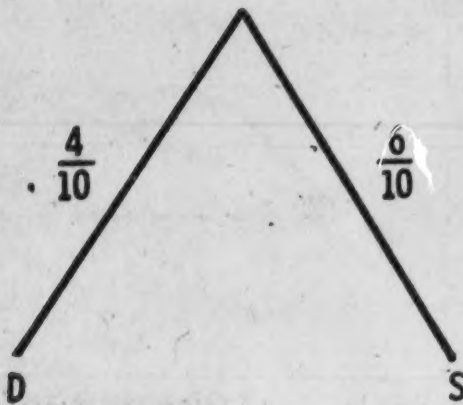
10 possible ways

Turn the page

There are 4 chances out of 10 of getting different colors. The probability of getting different colors is $\frac{4}{10}$.

There are 6 chances out of 10 of getting the same color. The probability of getting the same color is $\frac{6}{10}$.

The probabilities are shown in the following picture:



"D" means "different colors "

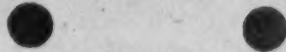
"S " means "same color "

Name _____

P02

*

Here are two red and two blue marbles.

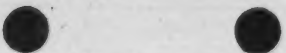


In the following picture show all possible ways of taking two marbles of the same color (both blue or both red).



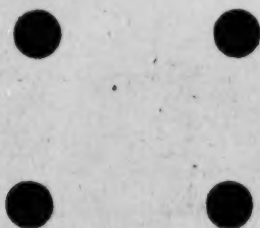
How many ways? _____

In the following picture show all possible ways of taking two marbles of different colors.



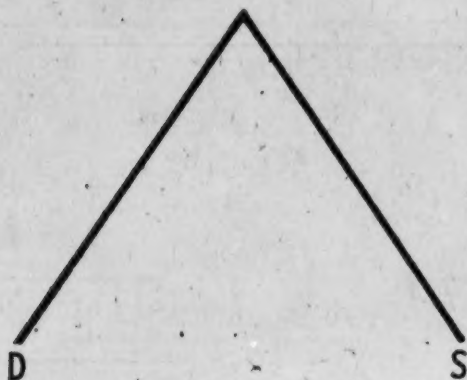
How many ways? _____

In the following picture show all possible ways of taking two marbles.



How many ways? _____

Write the probabilities next to the two lines in this picture:



Are you more likely to get different colors or the same color? _____

Which has the larger probability, D or S? _____

Name _____

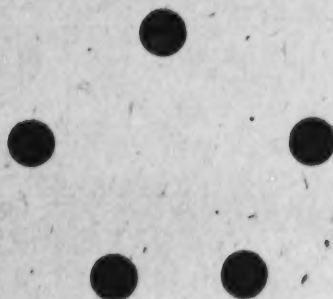
P03

Here are two red and three blue marbles.

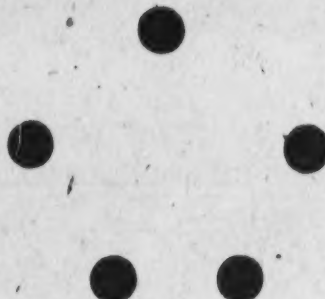


In the following picture show

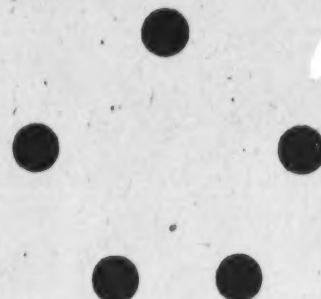
all possible ways of
taking two marbles
of different colors.



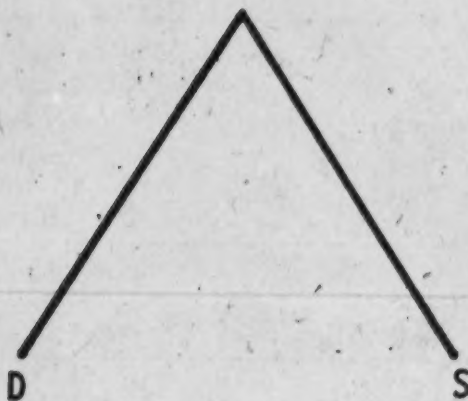
all possible ways of
taking two marbles
of the same color.



all possible ways of
taking two marbles.



Write the probabilities next to the two lines in this picture:



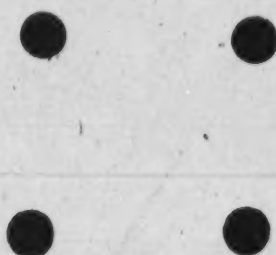
Are you more likely to get different colors or the same color? _____

Name _____

P04

**

Here are one red, one black and two blue marbles.



In the picture above show all possible ways to take two marbles.

How many ways can you take two marbles of the same color? _____

How many ways can you take two marbles of different colors? _____

What is the total number of ways of taking two marbles? _____

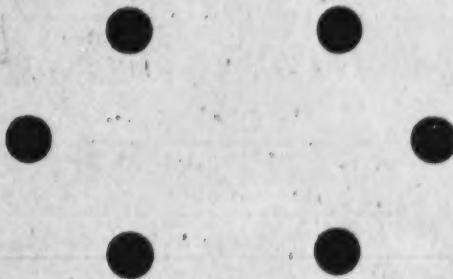
Write the probabilities next to the two lines in this picture:



Name _____

P05******

Here are one red and five blue marbles.



You mix them up and take two marbles without looking. Would you bet they are the same color or different colors? _____

What is the probability of getting different colors? _____

What is the probability of getting the same color? _____

Write the probabilities next to the two lines in this picture:



Name _____

P06

You have five blue and two red marbles. You mix them up and take two marbles without looking. Would you bet they are the same color or different colors? _____

What is the probability of getting different colors? _____

What is the probability of getting the same color? _____

(If you wish you may use the blank space below to draw any pictures or to make any calculations that may help you to answer the questions above.)

Name _____

P07*******

You have five blue marbles and three red marbles. You mix them up and take two marbles without looking. Would you bet they are the same color or different colors? _____

What is the probability of getting different colors? _____

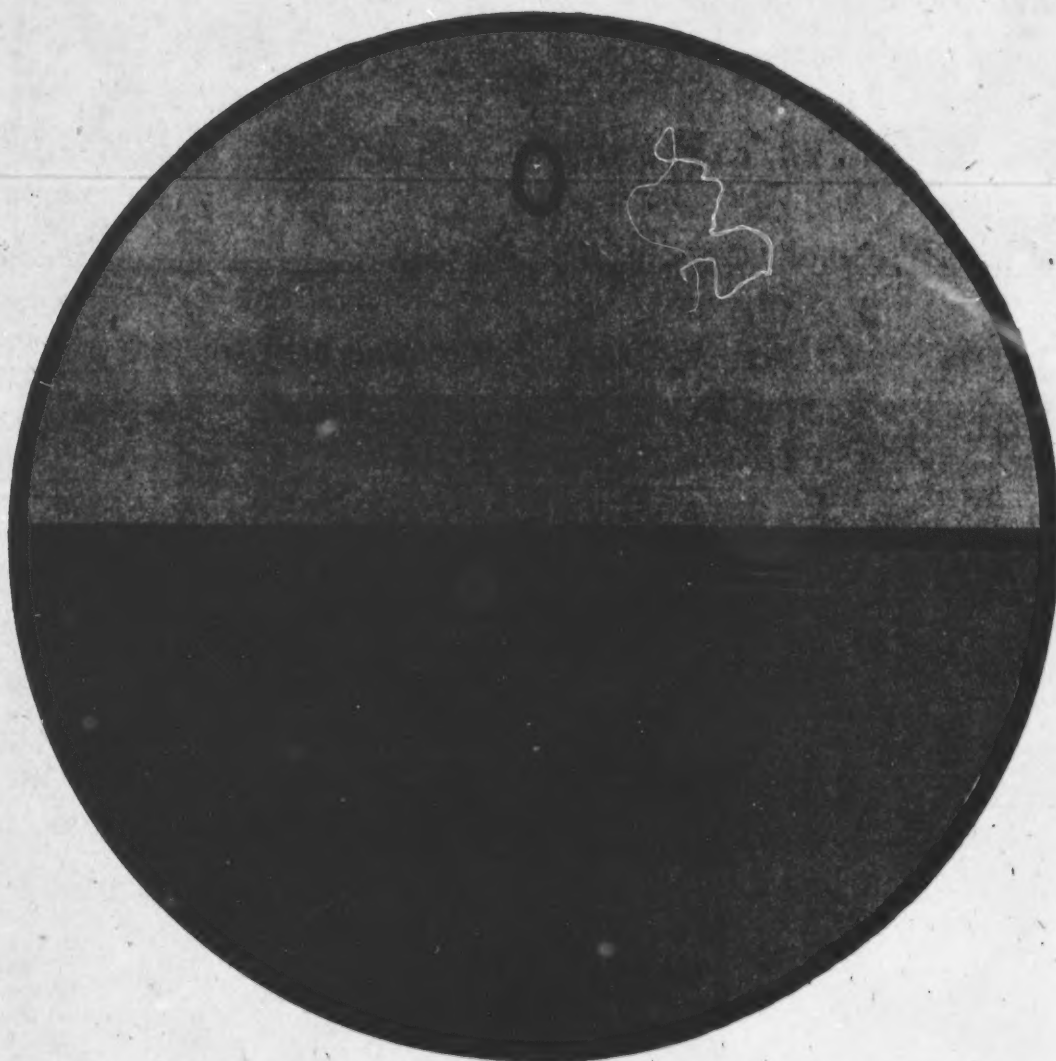
What is the probability of getting the same color? _____

(If you wish you may use the blank space below to draw any pictures or to make any calculations that may help you to answer the questions above.)

Name _____

P08

SPINNER SHEET



SPINNER FACE

FOR

"THE SQUARE TRAP"

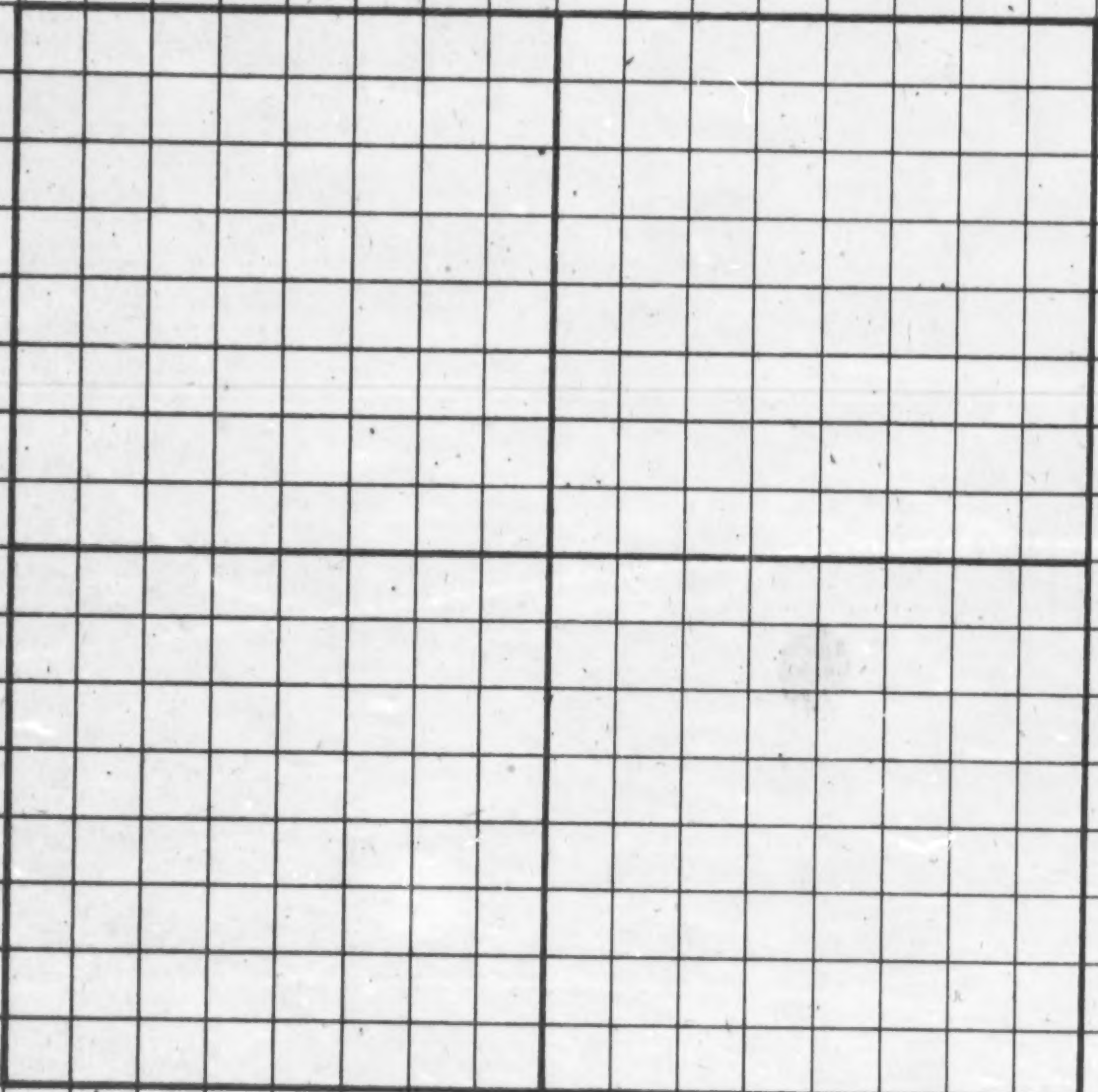
438

Name _____

G01

Name _____

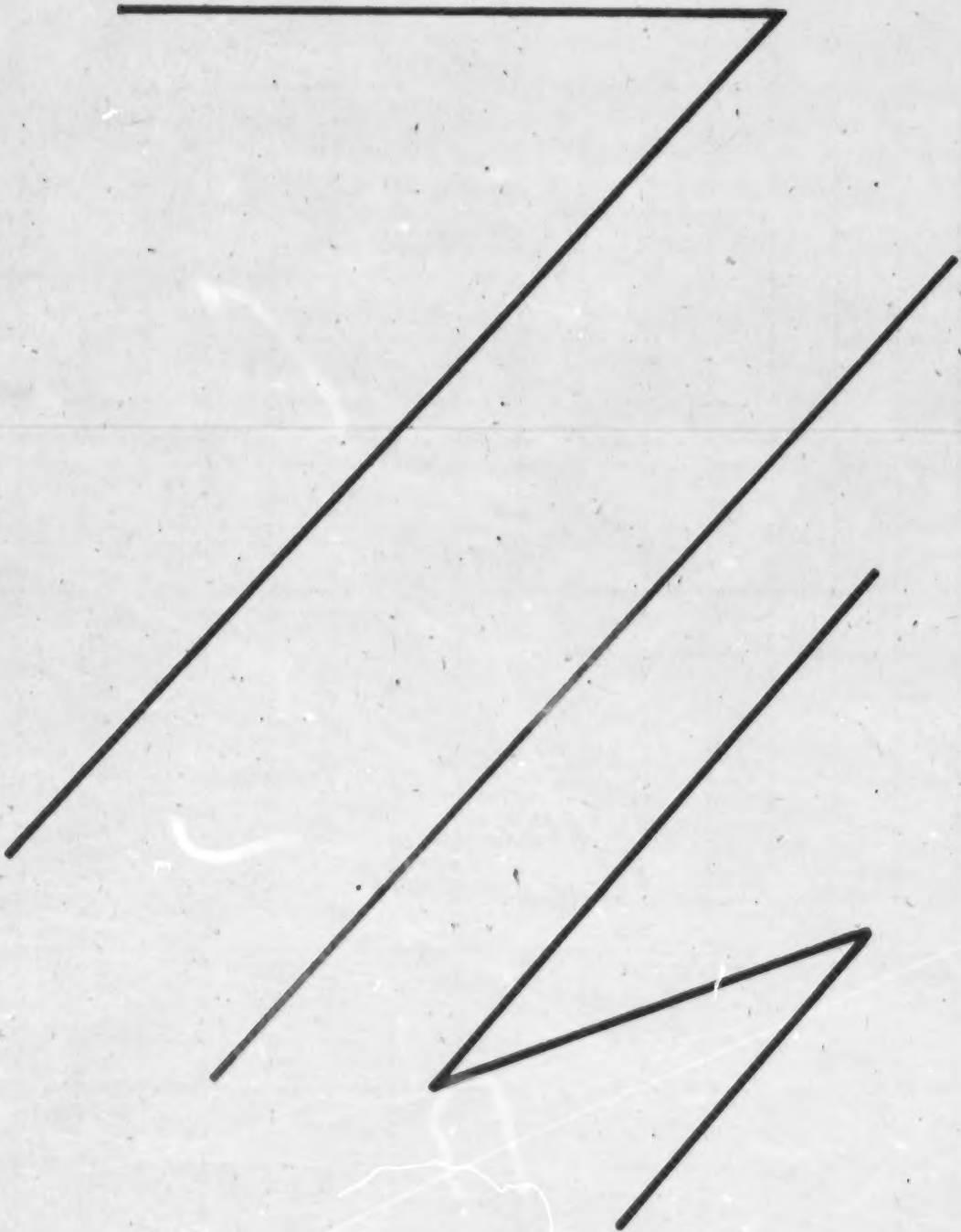
G02 *



Name _____

G03

*



Name _____

G04

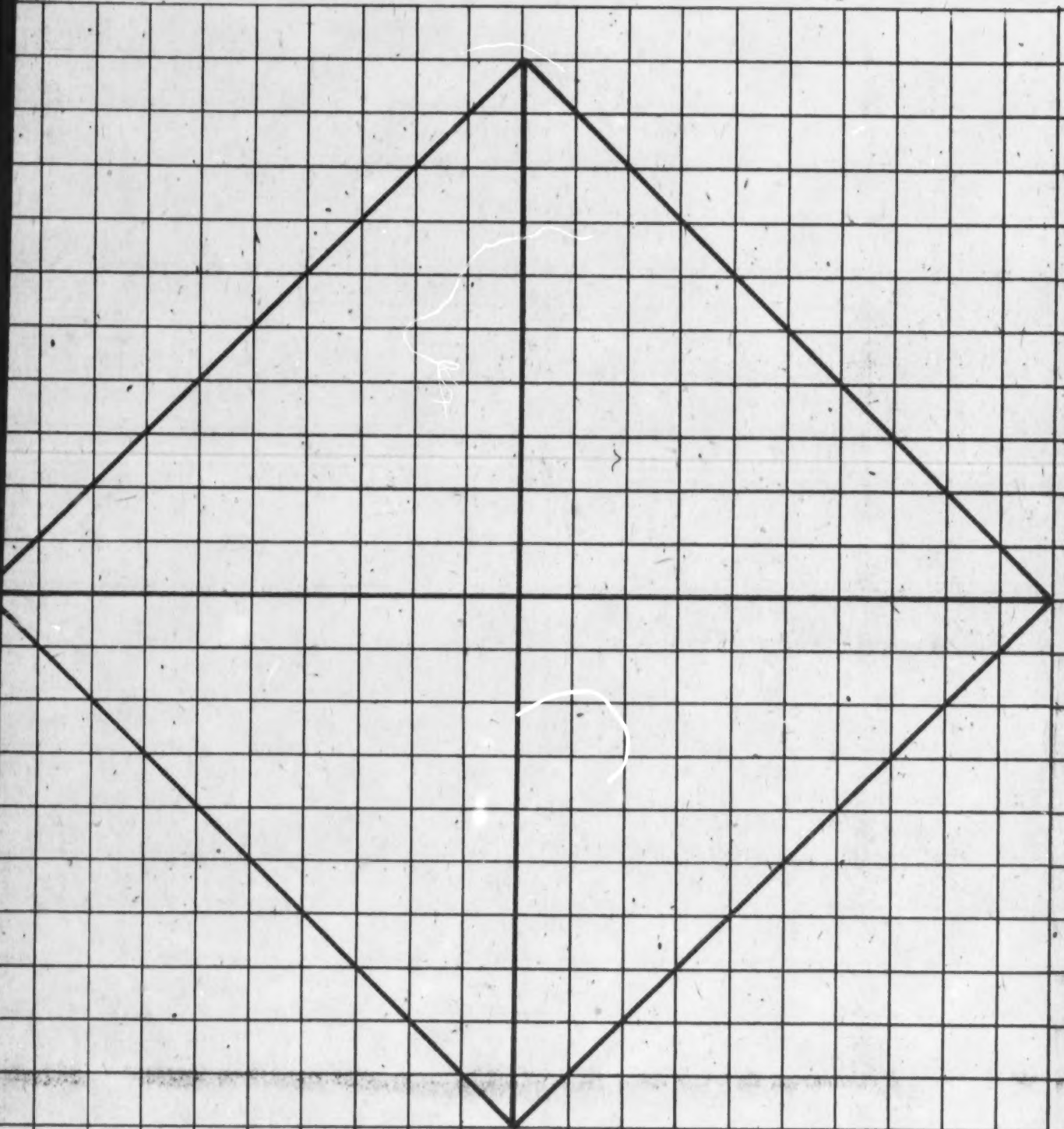
*

Name _____

G05

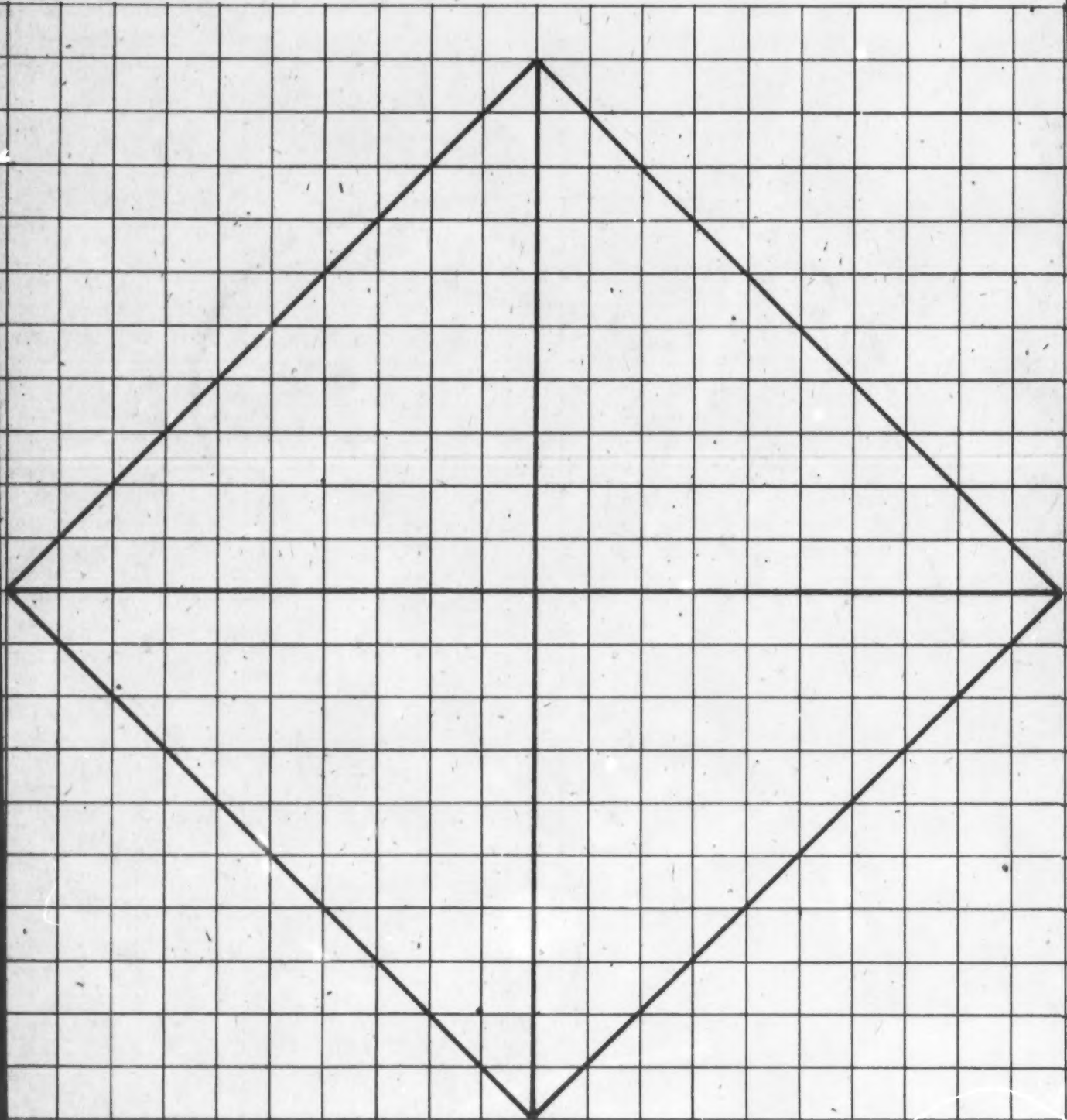
Name _____

G06



Name _____

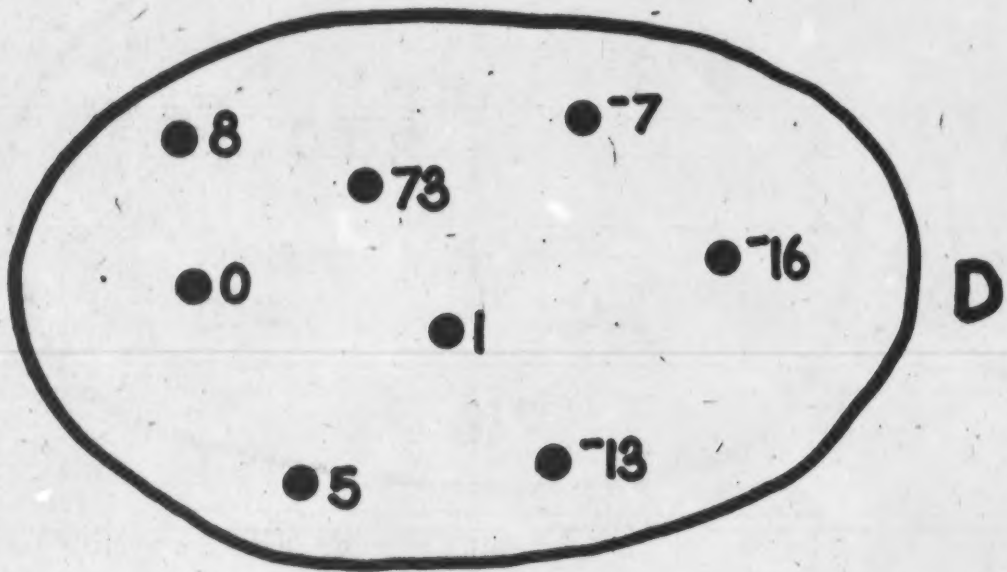
G07



Name _____

L 01 *

D is a set of exactly eight numbers. Below the string are some statements about D. Circle T if the statement is true and F if it is false.



T

☒ F1. No number in D is even.

T

☒ F2. All numbers in D are even.☒ T

F

3. At least two numbers in D are even.☒ T

F

4. At least three numbers in D are even.

T

☒ F5. Each number in D is either positive or negative.

T

☒ F6. At most one number in D is even.

T

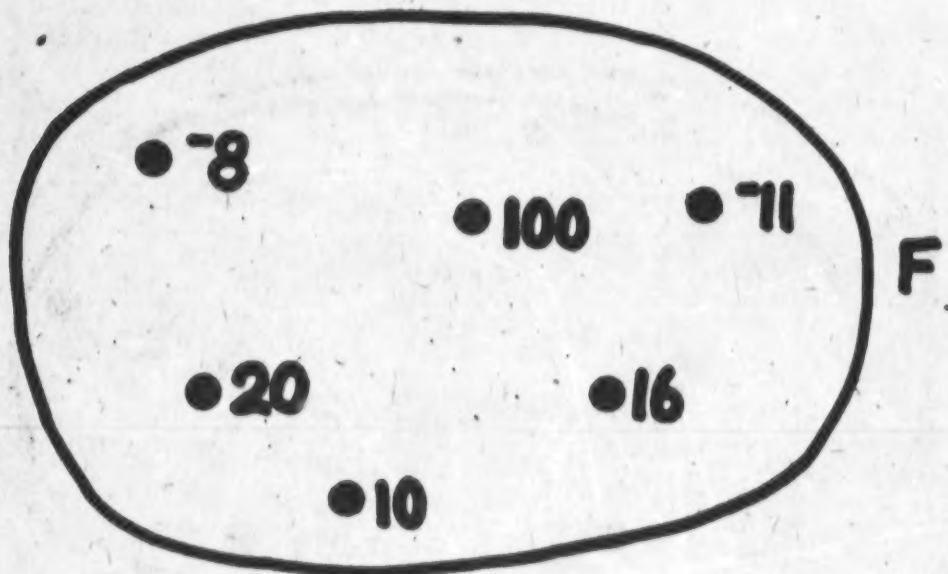
☒ F7. At most two numbers in D are even.

Name _____

L 02

**

F is a set of exactly six numbers. Below the string are some statements about F. Circle T if the statement is true and F if it is false.

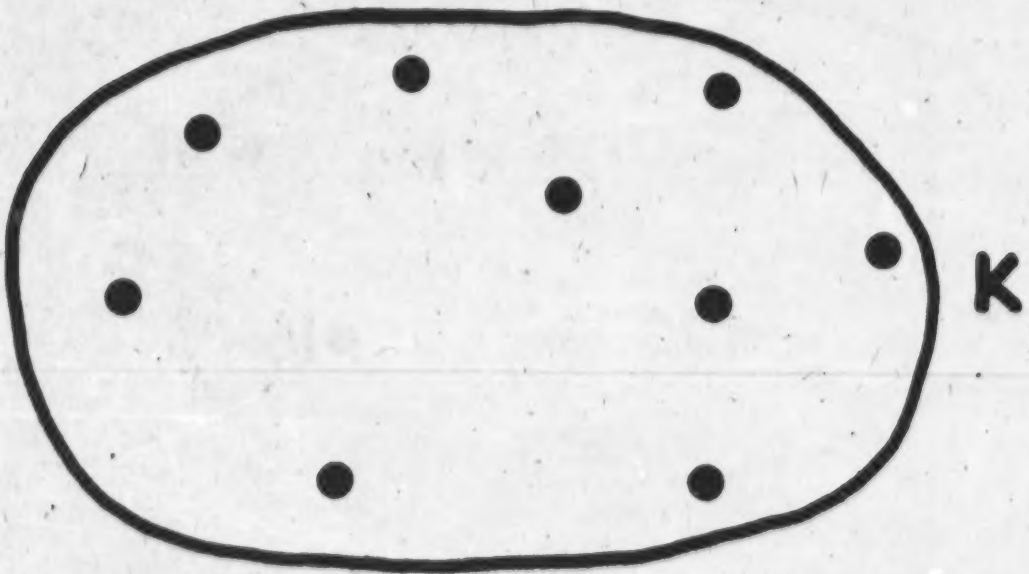


- | | | |
|---|---|---|
| T | F | 1. <u>Each</u> member of F is a multiple of 4. |
| T | F | 2. <u>No</u> member of F is a multiple of 4. |
| T | F | 3. <u>At least</u> one member of F is a multiple of 4. |
| T | F | 4. <u>At most</u> one member of F is a multiple of 4. |
| T | F | 5. <u>Exactly</u> four members of F are multiples of 4. |
| T | F | 6. <u>At least</u> three members of F are negative. |
| T | F | 7. <u>At most</u> three members of F are negative. |
| T | F | 8. <u>Every</u> even number in F is a multiple of 4. |

Name _____

L03

K is a set of exactly nine numbers.



Label the dots so that:

each number in K is even

and

at least two numbers in K are negative

and

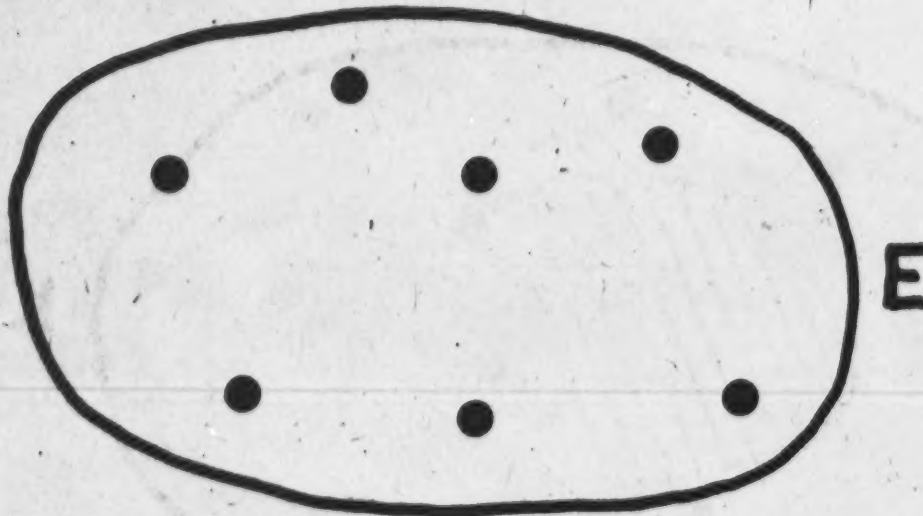
at most one number in K is a multiple of 5.

Name _____

L 04

**

E is a set of exactly seven numbers.



Label the dots so that:

each number in E is a multiple of 3

and

at most two numbers in E are larger than 7

and

at least four numbers in E are odd.

Name _____

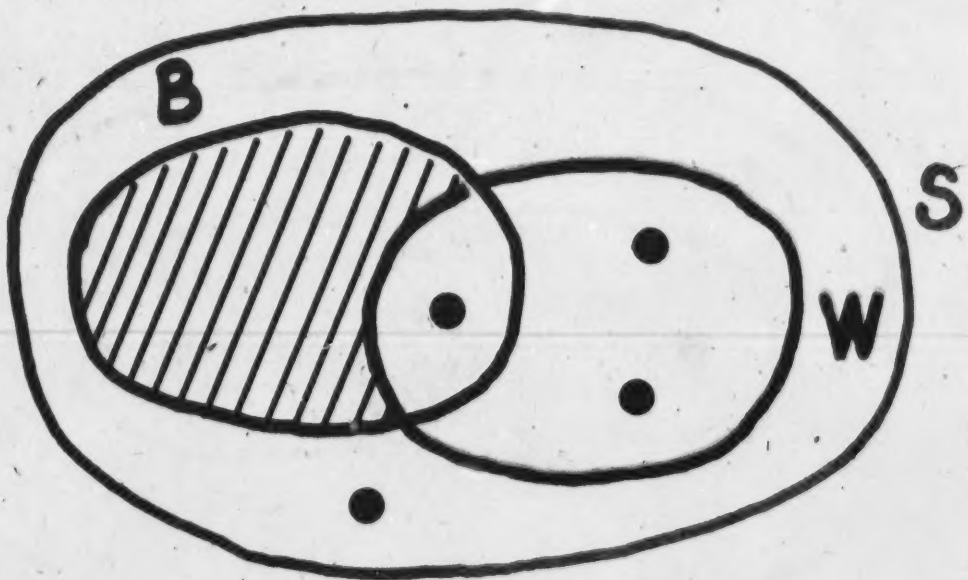
L 05

*

S is for all the students in Max's school.

B is for all the boys in Max's school.

W is for all the students in Max's school who wear glasses.



Circle: TRUE or FALSE

T F

1. Each boy in Max's school wears glasses.

T F

2. Each student in Max's school who wears glasses is a girl.

T F

3. At least one girl in Max's school wears glasses.

T F

4. At most one girl in Max's school wears glasses.

T F

5. At least one girl in Max's school does not wear glasses.

Name _____

L 06

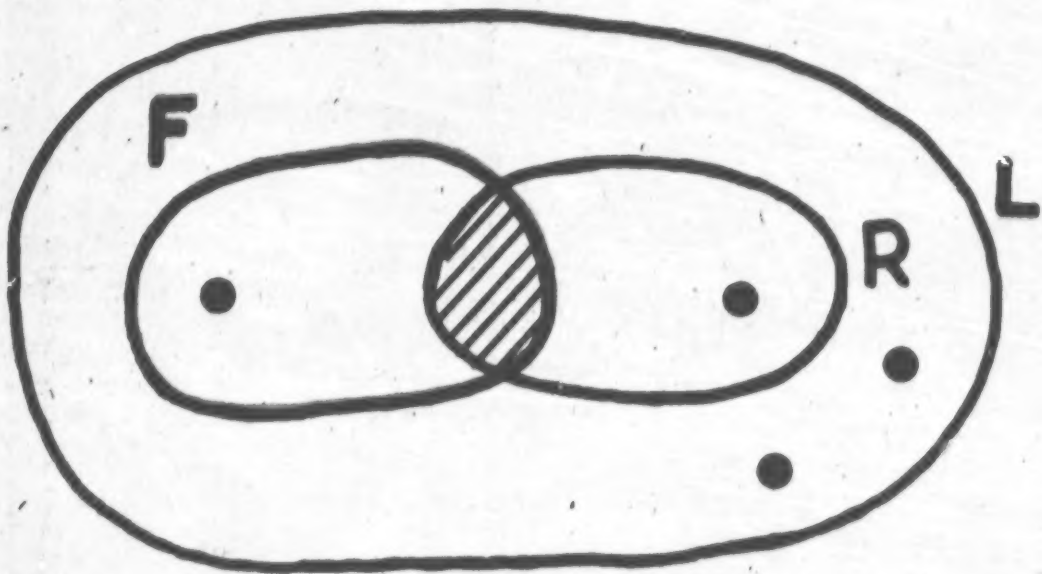
**

L is for all the cars on our school's parking lot.

F is for all the Fords on this lot.

R is for all the red cars on this lot.

(Not all the dots are drawn.)



Circle: TRUE or FALSE

T F

T F

T F

T F

T F

T F

T F

T F

1. On this lot, each Ford is red.

2. On this lot, no Ford is red.

3. On this lot, at least one Ford is red.

4. On this lot, at most one Ford is red.

5. On this lot, at most one car is not a Ford.

6. On this lot, at least one car is not red.

7. On this lot, at most two cars are not red.

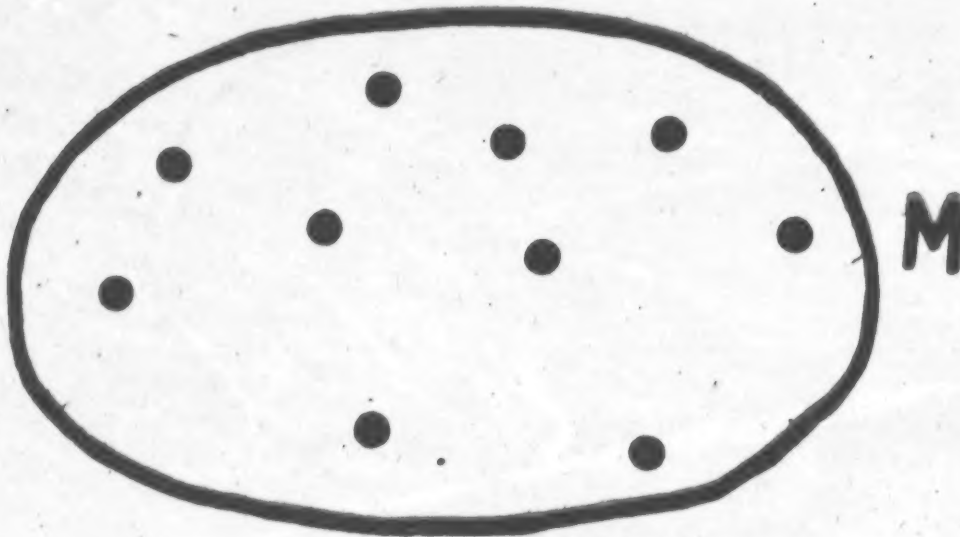
8. On this lot, at least one car is both not a Ford and not red.

Name _____

L 07

**

M is a set of exactly ten numbers.



Label the dots so that in M:

each number is between -30 and 20

and

no number is a multiple of 4

and

exactly three numbers are even

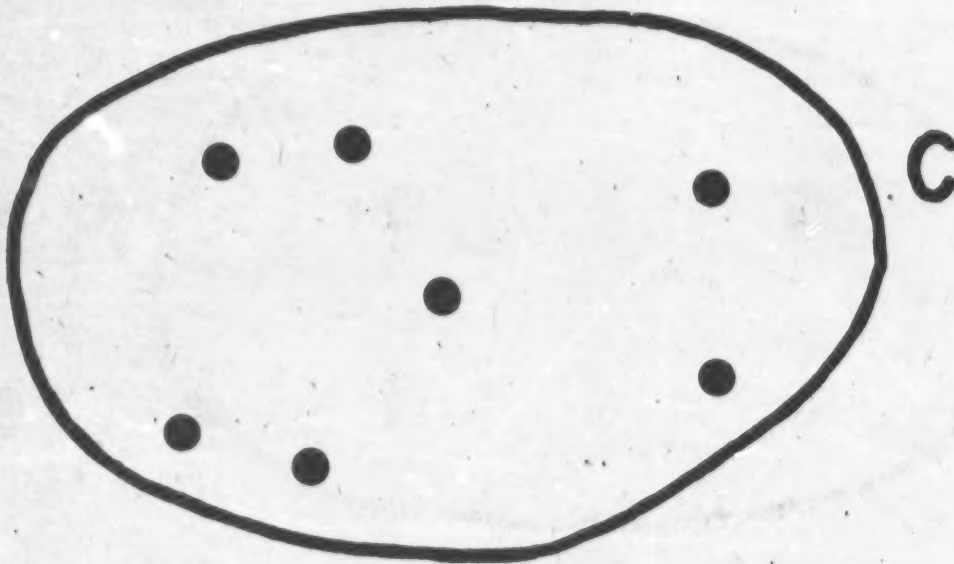
and

at most four numbers are positive.

Name _____

L 08

C contains exactly seven numbers.



Label the dots so that:

at least four numbers in C are multiples of 7

and

at least five numbers in C are multiples of 3

and

at most two numbers in C are even

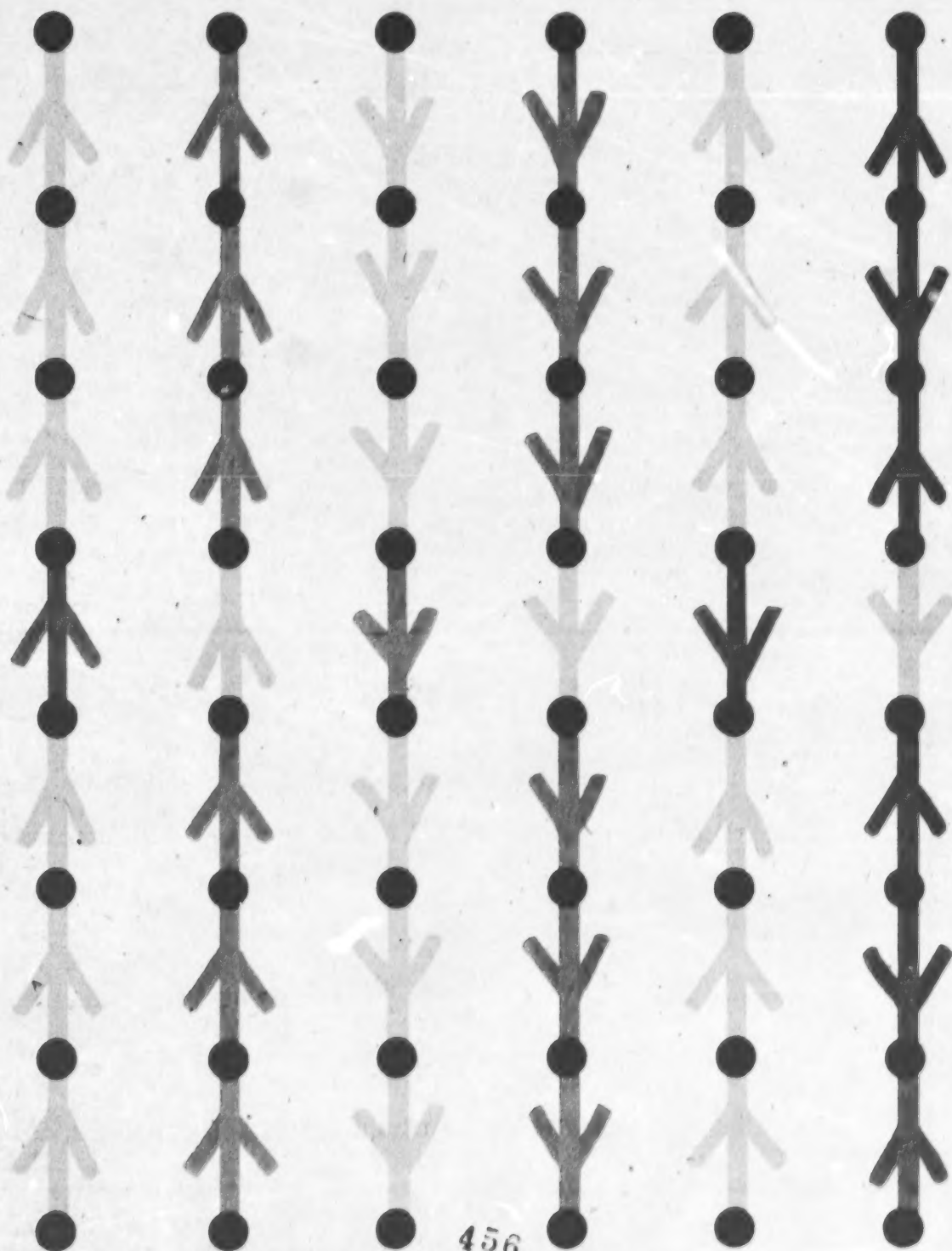
and

every number in C is a whole number smaller than 22.

Name _____

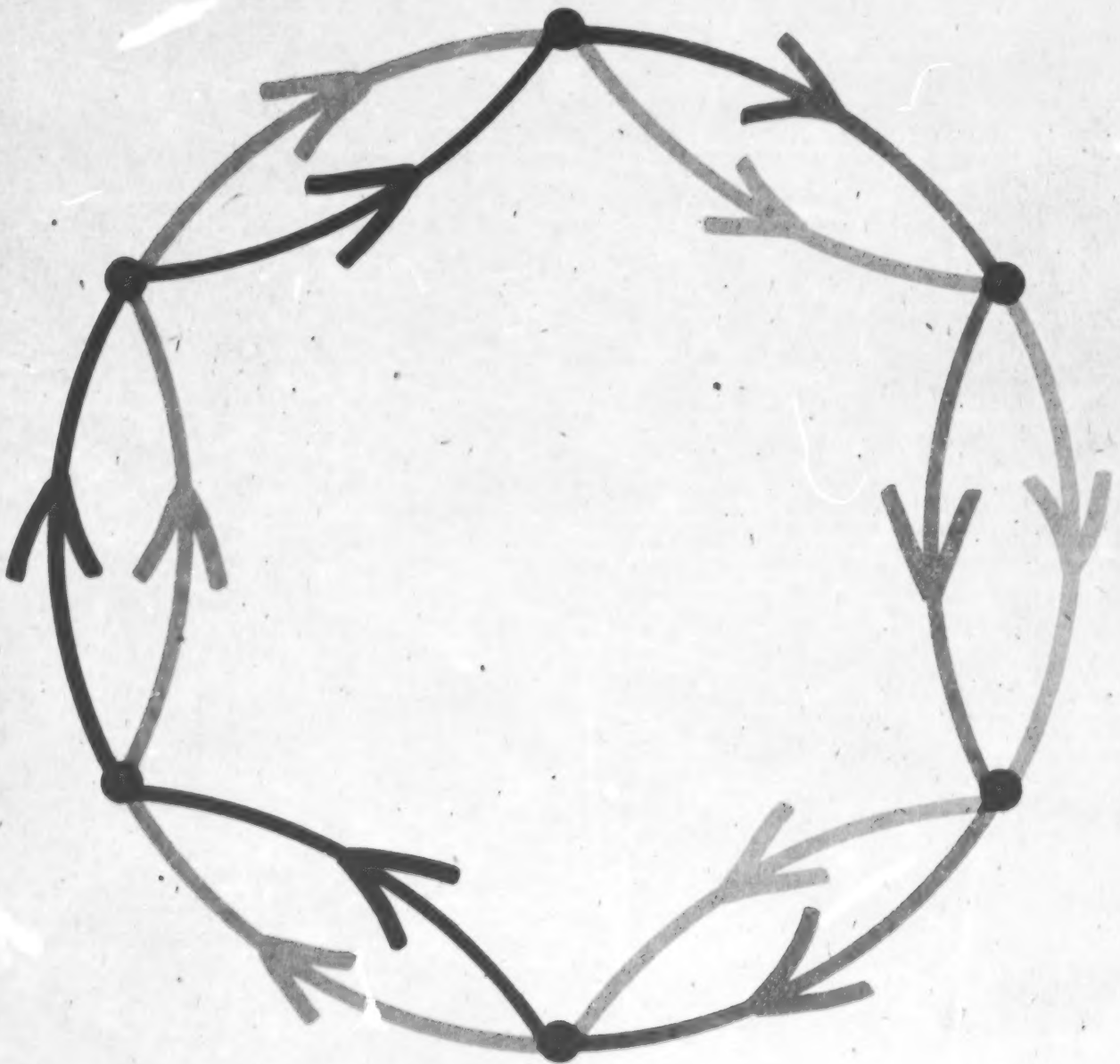
L 09

*



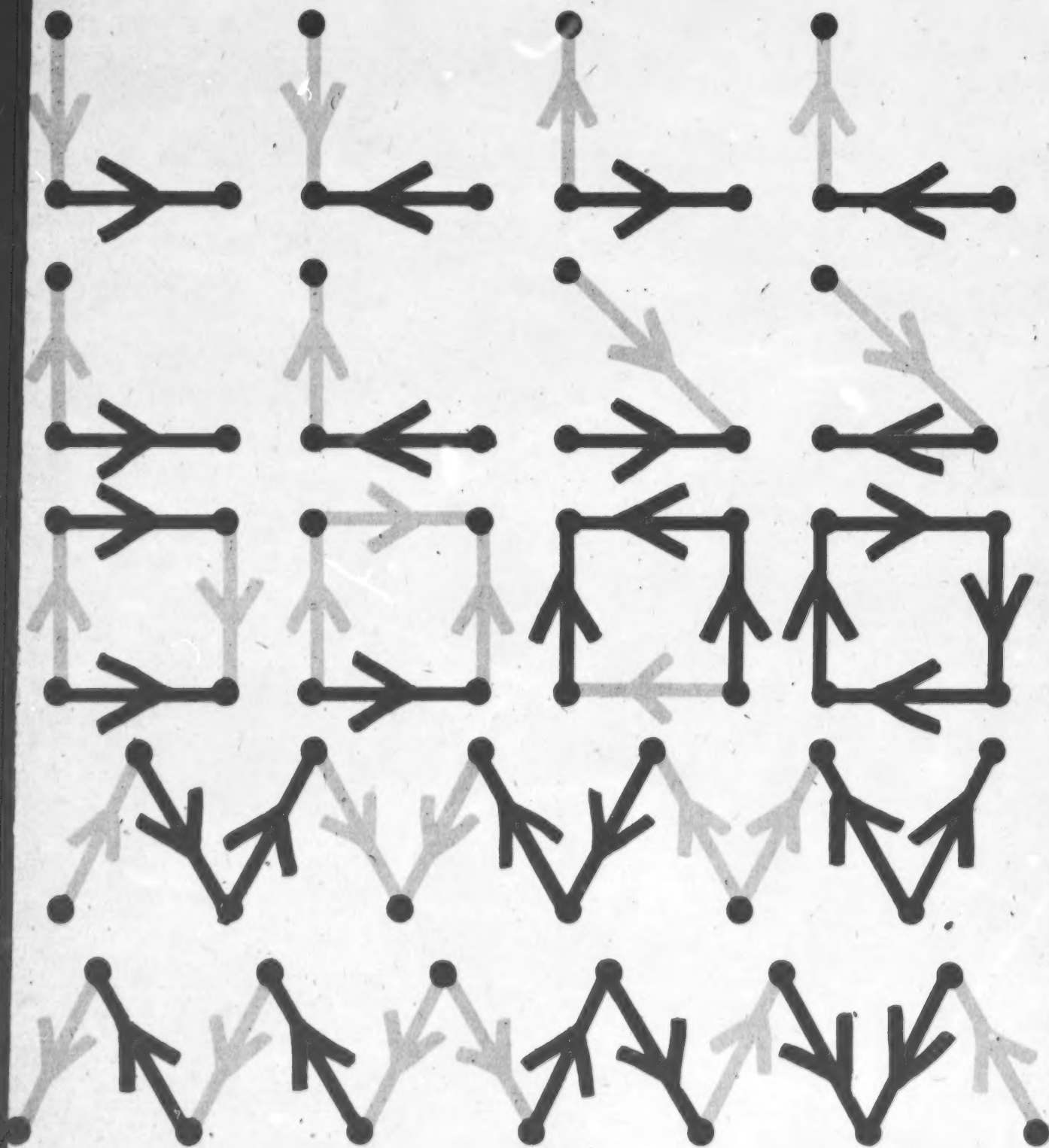
Name _____

L10



Name _____

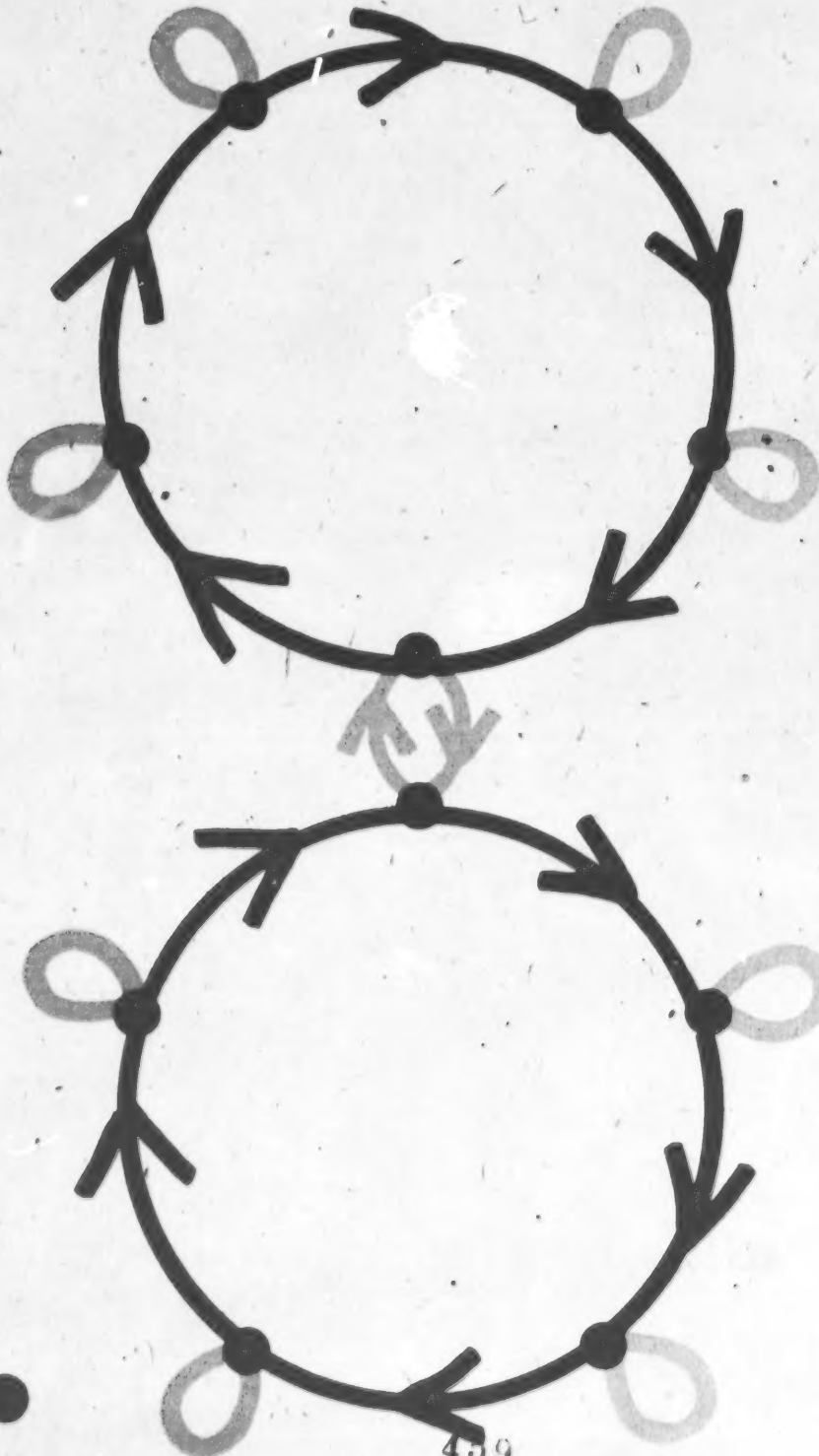
L 11



Name _____

L12

*

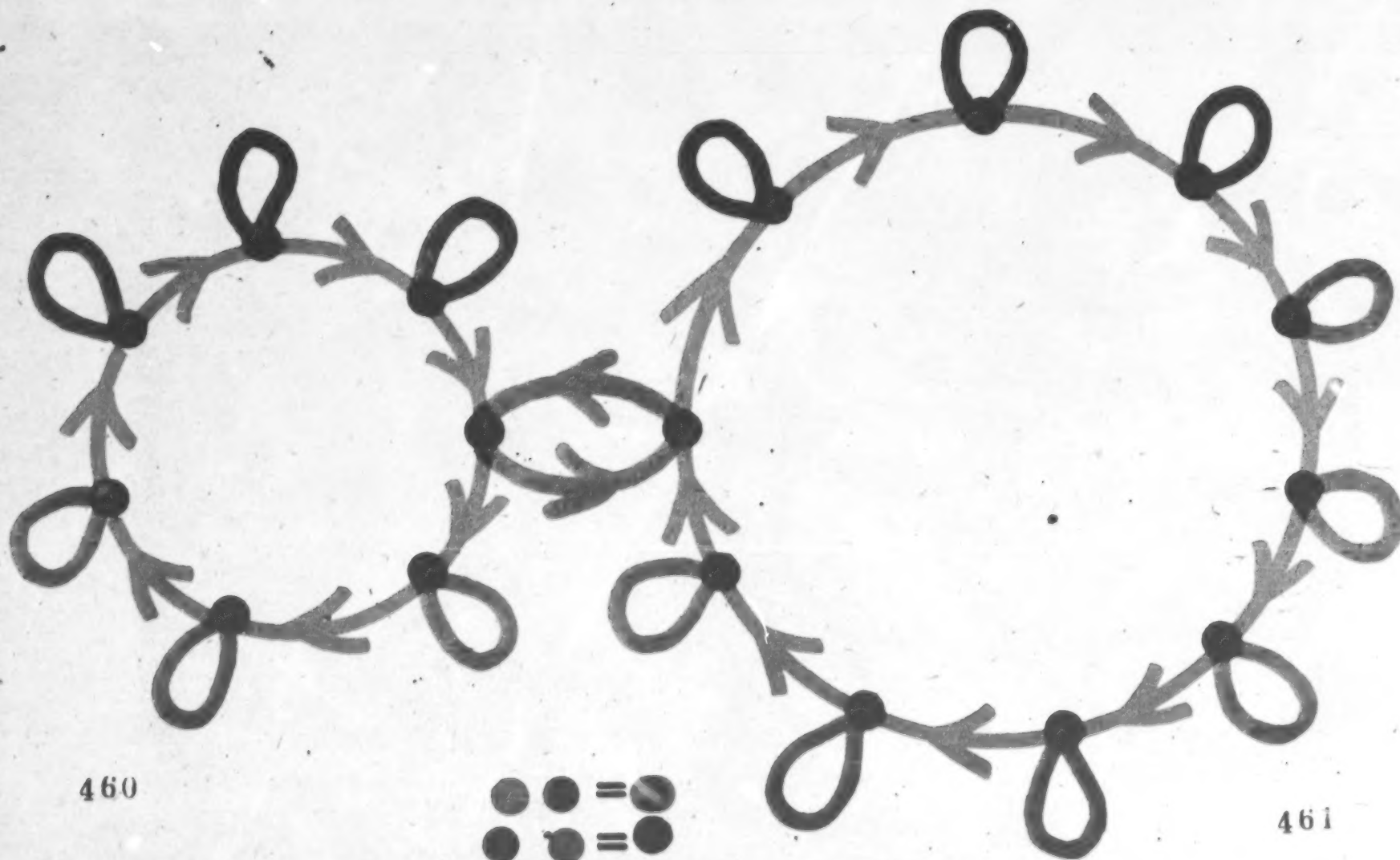


● ● = ●

Name _____

L13

**



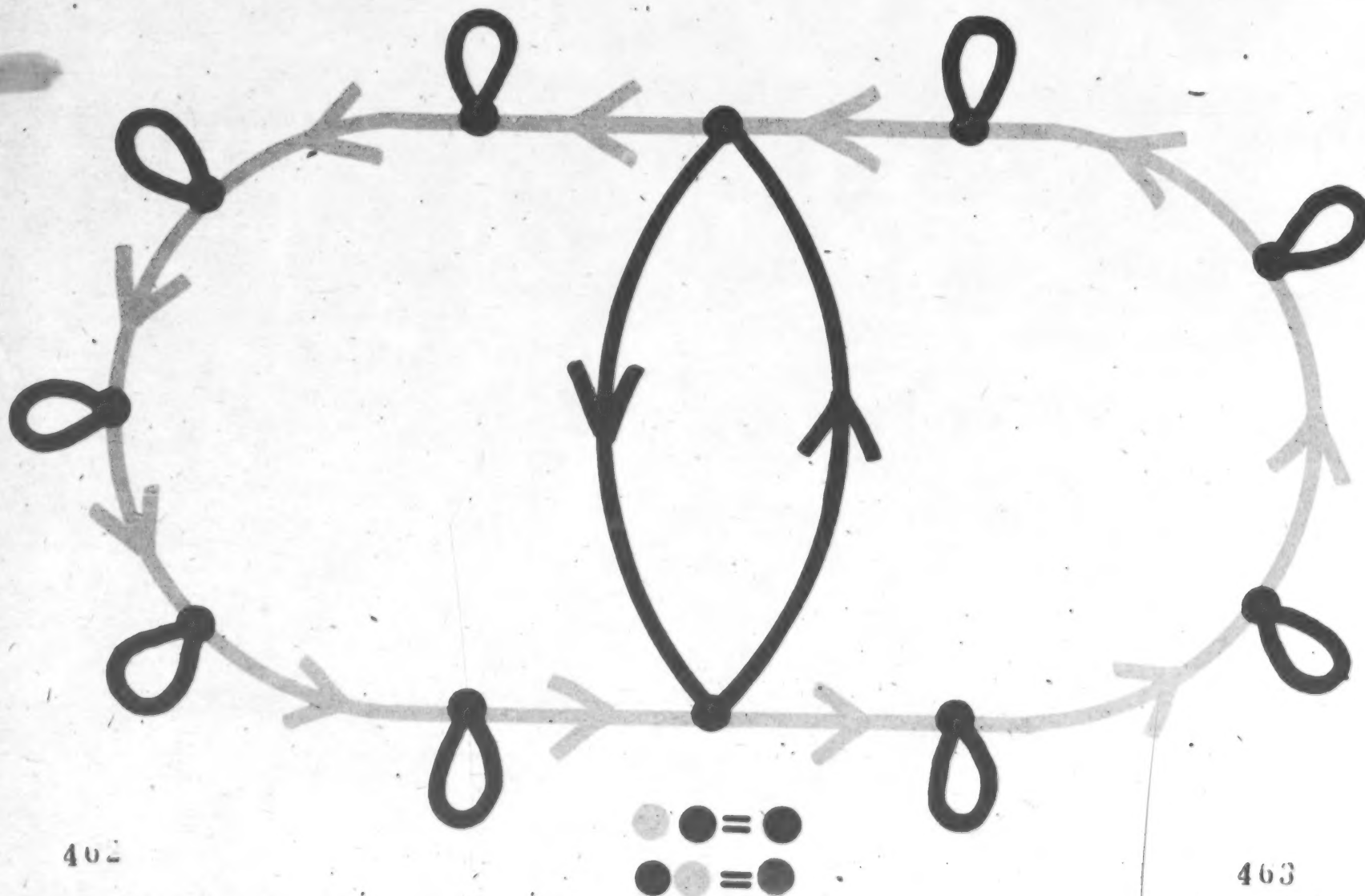
460

461

Name _____

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L14



402

463